Section 2.2, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete math at your school. Express each of the following sets in terms of A and B.

a. The set of sophomores taking discrete math at your school.

That's the intersection $A \cap B$.

b. The set of sophomores at your school who are not taking discrete math.

This is the difference $A - B$. It can also be expressed by intersection and complement $A \cap \overline{B}$.

c. The set of students at your school who either are sophomores or are taking discrete math.

The union $A \cup B$.

d. The set of students at your school who either are not sophomores or are not taking discrete math.

Literally, it's $\overline{A} \cup \overline{B}$. That's the same as $\overline{A \cap B}$.

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a. $A \cup B$. That's just B since every element of A is also an element of B.

b. $A \cap B$. That's just A in this case.

c. $A - B$. This is $\emptyset$ since there are no elements of A that aren't elements of B.

d. $B - A$. That's $\{f, g, h\}$, since those are the three elements in B that aren't in A.

6. Show the identity laws in Table 1. Besides the methods given here, you could use Venn diagrams or just explain things in words.

a. Show $A \cup \emptyset = A$. Use the fact that $x \in \emptyset$ is always false.

$$\begin{aligned} A \cup \emptyset &= \{x \mid x \in A \vee x \in \emptyset\} \\ &= \{x \mid x \in A\} = A \end{aligned}$$

7. Show the "domination" laws in Table 1.

b. Show $A \cup U = U$. Here, U is the universal set, i.e., the domain of discourse, so $x \in U$ is always true.

$$\begin{aligned} A \cup U &= \{x \mid x \in A \vee x \in U\} \\ &= \{x \mid x \in U\} = U \end{aligned}$$

16. Proofs involving subsets. There are various styles you can use to present your proofs. Here, the proofs given for parts **a** and **b** stick with sets, but the proof for part **e** uses elements. You can also use Venn diagrams, but be sure to explain what the diagrams say if you use them.

a. Prove $A \cap B \subseteq A$. This corresponds to the tautology $P \wedge Q \Rightarrow P$.

$$\begin{aligned} A \cap B &= \{x \mid x \in A \wedge x \in B\} \\ &\subseteq \{x \mid x \in A\} = A \end{aligned}$$

b. Prove $A \subseteq A \cup B$. This corresponds to the tautology $P \Rightarrow P \vee Q$.

$$\begin{aligned} A &= \{x \mid x \in A\} \\ &\subseteq \{x \mid x \in A \vee x \in B\} = A \cup B \end{aligned}$$

e. Prove $A \cup (B - A) = A \cup B$.

$$\begin{aligned} & x \in A \cup (B - A) \\ \iff & x \in A \vee x \in (B - A) \\ \iff & x \in A \vee (x \in B \wedge x \notin A) \\ \iff & (x \in A \vee x \in B) \wedge (x \in A \vee x \notin A) \\ \iff & x \in A \vee x \in B \\ \iff & x \in A \cup B \end{aligned}$$

Thus, $A \cup (B - A)$ and $A \cup B$ have the same elements, so they're equal.

23. Prove the distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

It's enough to say that it follows directly from the distributive law for propositional logic $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$. But if you like, you can prove it directly. Such a proof is almost identical to proving the corresponding logical distributive law; it just needs extra notation to make it set theoretical.

30. Can you conclude that $A = B$ when

a. $A \cup C = B \cup C$? No, C could cover their differences. Example: $A = \{a\}$, $B = \{b\}$, and $C = \{a, b\}$.

b. $A \cap C = B \cap C$? No, C could cut out their differences. Example: $A = \{a\}$, $B = \{b\}$, and $C = \emptyset$.

c. $A \cup B = B \cup C$ and $A \cap B = B \cap C$? No. This is harder to find a counterexample than the previous parts. But if both A and C are $\emptyset$, then both conditions are true, so B could be any set.

32. Find the symmetric difference of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

The elements that are in exactly one of the two sets are 2 and 5. Therefore, the symmetric difference is $\{2, 5\}$.

36. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

There isn't much to say. An element is in $A \oplus B$ iff it's in exactly one of the two sets A and B. That means it is in at least one of A or B, but not both, in other words, it is in $A \cup B$ but not in $A \cap B$, and that just says it's in $(A \cup B) - (A \cap B)$.

38b. Prove $(A \oplus B) \oplus B = A$.

One way is to use a truth table to cover all the cases.

$x \in A$	$x \in B$	$x \in A \oplus B$	$x \in (A \oplus B) \oplus B$
T	T	F	T
T	F	T	T
F	T	T	F
F	F	F	F

Since the column under A and the column under $(A \oplus B) \oplus B$ are the same, therefore the two sets are equal.

40. Is symmetric difference associative?

Yes, and you can see that most easily using a truth table for all eight cases.

In the process of proving this, you'll discover that $A \oplus B \oplus C$ consists of elements that belong to either 0 or 2 of the sets A, B, and C. More generally, the continued symmetric difference $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ consists of those elements that belong to an even number of the sets $A_1, A_2, \ldots, A_n$.

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