Section 2.3, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

2. Determine whether f is a function from $\mathbf{Z}$ to $\mathbf{R}$ if

a. $f(n) = \pm n$. This is not a function since functions are single valued, and this description is double valued.

b. $f(n) = \sqrt{n^2+1}$. Sure, this describes a function from $\mathbf{Z}$ to $\mathbf{R}$.

c. $f(n) = 1/(n^2-4)$. Not quite a function defined on $\mathbf{Z}$ since it's not defined at $n = \pm 2$.

4. Find the domain and range of the following functions.

a. The function that assigns to each nonnegative integer its last digit. Let's assume that the integers are denoted decimally. Then the domain is $\mathbf{Z}$, and the range is the set $\{0,1,2,3,4,5,6,7,8,9\}$.

b. The function that assigns the next largest integer to a positive integer. It explicitly says that the function is only defined for positive integers, so the domain is $\{n \in \mathbf{Z} \mid n \geq 1\}$. The range includes only integers 2 or greater $\{n \in \mathbf{Z} \mid n \geq 2\}$.

c. The function that assigns to a bitstring the number of one bits in the string. The domain is the set of bitstrings, sometimes denoted $\{0,1\}^*$, and the range is the natural numbers $\mathbf{N} = \{n \in \mathbf{Z} \mid n \geq 0\}$. (Note that 0 is included in the range because some bitstrings have no 1s.)

d. The function that assigns to a bitstring the number of bits in the string. Same domain and range as in 4c. (Note that 0 is included in the range because the empty string counts as a string.)

8. On ceiling and floor functions.

a. $\lfloor 1.1 \rfloor$. The floor of 1.1 is 1, the largest integer less than or equal to 1.1.

b. $\lceil 1.1 \rceil$. The ceiling of 1.1 is 2, the smallest integer greater than or equal to 1.1.

c. $\lfloor -0.1 \rfloor$. The floor of -0.1 is -1, the largest integer less than or equal to -0.1.

d. $\lceil -0.1 \rceil$. The ceiling of -0.1 is 0, the smallest integer greater than or equal to -0.1.

e. $\lceil 2.99 \rceil$. This is 3.

f. $\lceil -2.99 \rceil$. This is -2, not -3.

g. $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$. This is $\lfloor \frac{1}{2} + 1 \rfloor$, which equals 1.

h. $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$. This equals $\lceil 0 + 1 + \frac{1}{2} \rceil$, which is 2.

10. Determine whether each of the following functions from $\{a,b,c,d\}$ to itself is one-to-one.

a. $f(a) = b, f(b) = a, f(c) = c, f(d) = d$. Yep, it's one-to-one alrighty.

b. $f(a) = b, f(b) = b, f(c) = d, f(d) = c$. Nope, f of two different things equals b.

c. $f(a) = d, f(b) = b, f(c) = c, f(d) = d$. Nope, f of two different things equals d.

28. Let $f(x) = 2x$. What is

a. $f(\mathbf{Z})$? That's the set of even integers $\{n \in \mathbf{Z} \mid \exists m \in \mathbf{Z}, n = 2m\}$. There are various shorthand versions in common use, for instance $\{2m \in \mathbf{Z} \mid m \in \mathbf{Z}\}$. Even more abbreviated is the expression $2\mathbf{Z}$.

b. $f(\mathbf{N})$? Since the natural numbers $\mathbf{N}$ is the set of nonnegative integers, the image of f is the set of nonnegative even integers, written either $\{n \in \mathbf{N} \mid \exists m \in \mathbf{N}, n = 2m\}$ or $\{2m \in \mathbf{N}\}$ where m is implicitly a natural number.

c. $f(\mathbf{R})$? That's $\mathbf{R}$ since every real number has a half.

32. Find $f \circ g$ and $g \circ f$ where $f(x) = x^2+1$ and $g(x) = x+2$ are functions from $\mathbf{R}$ to $\mathbf{R}$.

$$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2+1 = x^2+4x+5.$$

$$(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2+1+2 = x^2+3.$$

Note that these are not the same; usually, $f \circ g$ does not equal $g \circ f$.

36. Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

a. $f(S \cup T) = f(S) \cup f(T)$.

There are many proofs that you can come up with. Here's one that shows that an element $y \in B$ belongs to the left hand side of the equation iff it belongs to the right hand side. I'm being a little more careful in this proof than necessary to show each step. Also, I made the proof entirely symbolic, but it would be easier to follow if it were said in words.

$$\begin{aligned}
& y \in f(S \cup T) \\
\iff & \exists x{\in}A\,(x{\in}S \cup T \wedge y{=}f(x)) \\
\iff & \exists x{\in}A\,((x{\in}S \vee x \in T) \wedge y{=}f(x)) \\
\iff & \exists x{\in}A\,((x{\in}S \wedge y{=}f(x)) \vee (x{\in}T \wedge y{=}f(x))) \\
\iff & (\exists x{\in}A\,(x{\in}S \wedge y{=}f(x))) \vee (\exists x{\in}A\,(x{\in}T \wedge y{=}f(x))) \\
\iff & y \in f(S) \vee y \in f(T) \\
\iff & y \in f(S) \cup f(T)
\end{aligned}$$

b. $f(S \cap T) \subseteq f(S) \cap f(T)$.

Here's a proof that shows if $y \in A$ belongs to the first set, then it belongs to the second one.

$$\begin{array}{cl} & y \in f(S \cap T) \\ \iff & \exists x{\in}A\,(x{\in}S \cap T \wedge y{=}f(x)) \\ \iff & \exists x{\in}A\,(x{\in}S \wedge x{\in}T \wedge y{=}f(x)) \\ \implies & (\exists x{\in}A\,(x{\in}S \wedge y{=}f(x))) \wedge (\exists x{\in}A\,(x \in T \wedge y{=}f(x))) \\ \iff & y \in f(S) \wedge y \in f(T) \\ \iff & y \in f(S) \cap f(T) \end{array}$$

Note that one implication can't be reversed, so this proof doesn't show that $f(S) \cap f(T) \subseteq f(S \cap T)$, which isn't always true anyway.

38. Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = x^2$. Find

a. $f^{-1}(\{0\})$. The only number whose square is 0 is 0, so the inverse image of the set $\{0\}$ is just $\{0\}$.

b. $f^{-1}(\{x \,|\, 0 < x < 1\})$. What numbers' squares lie between 0 and 1? Answer: $(-1, 0) \cup (0, 1)$.

c. $f^{-1}(\{x \,|\, x > 4\})$. Answer $(-\infty, -2) \cup (2, \infty)$.

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