Math 114 Discrete Mathematics

Section 6.4, selected answers

D Joyce, Spring 2018

1. What is the expected number of heads that come up when a fair coin is flipped five times?

Of course, you expect that half will be heads, so the answer ought to be 2.5. The real question is: how does our definition work to give this answer. The definition is

$$E(X) = \sum_{s \in S} p(s)X(s),$$

where each outcome s is one of the 32 different sequences of 5 coin flips, $p(s)$ is the probability of that outcome, and $X(s)$ is the number of heads in that outcome. Since the coin is fair, each probability $p(s)$ equals $\frac{1}{32}$. Now, for 1 outcome, namely TTTTT, X is 0; for 5 outcomes, $X = 1$; for 10, $X = 2$; for 10, $X = 3$; for 5, $X = 4$; and for 1, $X = 5$. Therefore, the expectation $E(X)$ is equal to

$$1 \cdot \tfrac{1}{32} \cdot 0 + 5 \cdot \tfrac{1}{32} \cdot 1 + 10 \cdot \tfrac{1}{32} \cdot 2 + 10 \cdot \tfrac{1}{32} \cdot 3 + 5 \cdot \tfrac{1}{32} \cdot 4 + 1 \cdot \tfrac{1}{32} \cdot 5.$$

That comes out to be 2.5.

See the answer to the next exercise for an easier way to justify this.

2. What is the expected number of heads that come up when a fair coin is flipped ten times?

Of course, the answer is again $\frac{1}{2}$ of 10, that is, 5. A much shorter argument can be made by using linearity of expectation. Since X, the number of heads, is the sum $X_1 + X_2 + X_3 + X_4 + X_5$ of five random variables, where X_i is the expected number of heads for the single i^{th} coin flip, that is X_i is the probability of getting H on the i^{th} flip, therefore, by the linearity of expectation,

$$\begin{aligned} & E(X) \\ = \; & E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) \\ = \; & \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{10}{2} = 5. \end{aligned}$$

4. A coin is biased so that the probability of H is 0.6. What is the expected number of heads when it's flipped 10 times?

You can use an argument like that given in 2 using linearity of expectation, above, to see that the expectation is 10 times the expectation of H on one flip, that is, 10 times 0.6. So the answer is 6.

6. What's the expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \ldots, 50\}$ and the purchaser wins nothing otherwise?

We'll need to compute the probability of winning. There are $\binom{50}{6}$ possible tickets, only one of which is winning, so the the probability of winning is $1/\binom{50}{6}$. The probability of losing is 1 minus that. When the purchaser wins, there is a net gain of \$9,999,999, but when the purchaser loses, then there is a net loss of \$1. Thus, the expected value is

$$\begin{aligned} & 9999999 \cdot P(\text{win}) - 1 \cdot P(\text{lose}) \\ = \; & 9999999/\tbinom{50}{6} - \left(1 - 1/\tbinom{50}{6}\right) \\ = \; & 10000000/\tbinom{50}{6} - 1 \\ = \; & 1000000/15890700 - 1 = .629 - 1 = -.371 \end{aligned}$$

Thus, you expect to lose on average about 37 cents of your dollar each time you play this lottery.

(This would be a very bad lottery to play. Real lotteries aren't any better. Less than half the money spent on lotteries goes back to the purchasers.)

12. Suppose that we roll a die until a 6 comes up.

a. What is the probability that we roll the die n times?

Let X be the number of times we roll the die to get the first 6.

$P(X{=}1) = \frac{1}{6}$

$P(X{=}2) = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{6^2}$

$P(X{=}3) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{5^2}{6^3}$

In general,

$$P(X{=}n) = \frac{5^{n-1}}{6^n}.$$

This is an example of a geometric distribution with the parameter $p = \frac{1}{6}$.

b. What is the expected number of times we roll the die?

The theorem about geometric distributions in the textbook says the expectation is

$$E(X) = \frac{1}{1-p} = 6.$$

16. Let X and Y be random variables that give the number of heads and tails when two coins are flipped. Show they're not independent random variables.

Although there are lots of other ways to show that they're independent, it's enough to show that $E(XY) \neq E(X)E(Y)$. First note that $E(X)$ and $E(Y)$ are both 1. Next to compute $E(XY)$. With probability $\frac{1}{4}$, $X = 2$ and $Y = 0$, so $XY = 0$; with probability $\frac{1}{2}$, $X = 1$ and $Y = 1$, so $XY = 1$; and with probability $\frac{1}{4}$, $X = 0$ and $Y = 2$, so $XY = 0$. Thus

$$E(XY) = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{1}{2}.$$

Therefore, $E(XY) = \frac{1}{2}$, but $E(X)E(Y) = 1$. Since $\frac{1}{2} \neq 1$, therefore X and Y are not independent random variables.

Math 114 Home Page at `http://math.clarku.edu/~djoyce/ma114/`