CLARK UNIVERSITY 1887

Second Test Answers
Math 120 Calculus I
October, 2013

Scale. 89–100 A, 78–88 B, 63–77 C.

1. [15] On implicit differentiation. The point $(1,0)$ lies in the curve $y^3 - y = 1.2(x^3 - x)$. Determine the slope of the line tangent to the curve at that point.

Start by taking the derivative of $y^3 - y = 1.2(x^3 - x)$ with respect to x. You'll get

$$3y^2\frac{dy}{dx} - \frac{dy}{dx} = 1.2(3x^2 - 1).$$

Solve for $\frac{dy}{dx}$ from that equation:

$$\frac{dy}{dx} = \frac{1.2(3x^2-1)}{(3y^2-1)}$$

That's the slope of the tangent line at a point (x, y) on the curve. When (x, y) is the point $(1,0)$ that is equal to

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{1.2(3-1)}{(0-1)} = -2.4.$$

2. [10] Graphs and derivatives.

a. [5] Sketch the graph $y = f(x)$ of a function for which is continuous everywhere, and differentiable everywhere except at $x = 4$.

There are lots of such graphs. The important feature is that there should be a "corner" in the graph when $x = 4$.

b. [5] Sketch the graph $y = f(x)$ of a function that is differentiable everywhere and whose derivative is 0 at $x = 3$ and $x = 5$ but nonzero everywhere else.

Again, there are lots of such graphs. There should be horizontal tangents of the graph at $x = 3$ and $x = 5$.

3. [15] Recall the definition of derivatives in terms of limits, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Use that definition to show that the derivative of $f(x) = 5 + \frac{1}{x}$ is $f'(x) = -\frac{1}{x^2}$. (Do not use any of the rules of differentiation, just the definition.)

Here are the steps if you don't leave any of them out. Some can be combined, of course.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(5+1/(x+h))-(5+1/x)}{h} \\
&= \lim_{h\to 0}\frac{5+1/(x+h)-5-1/x}{h} \\
&= \lim_{h\to 0}\frac{1/(x+h)-1/x}{h} \\
&= \lim_{h\to 0}\frac{x-(x+h)}{h(x+h)x} \\
&= \lim_{h\to 0}\frac{-h}{h(x+h)x} \\
&= \lim_{h\to 0}\frac{-1}{(x+h)x} \\
&= \frac{-1}{x^2}
\end{aligned}$$

4. [15] On logarithmic differentiation. The function $y = f(x) = (x^2+4)^x$ cannot be differentiated by the power rule since the exponent is not constant, and it can't be differentiated by the exponential rule since the base is not constant, but you can find its derivative with logarithmic differentiation. Find its derivative. Show your work, and write carefully. Express your answer $f'(x)$ in terms of x.

First take ln of the equation $f(x) = (x^2+4)^x$ to get

$$\ln f(x) = x\ln(x^2+4).$$

Then differentiate with respect to x. You'll need the chain rule and the product rule.

$$\frac{f'(x)}{f(x)} = \ln(x^2+4) + x\,\frac{2x}{x^2+4}.$$

Finally solve for $f'(x)$ to get

$$f'(x) = (x^2+4)^x \left(\ln(x^2+4) + \frac{2x^2}{x^2+4} \right).$$

5. [45; 9 points each part] Differentiate the following functions. Do not simplify your answers. Use parentheses properly.

a. $f(t) = 7t^4 - 4\sqrt{t} + 6 + \dfrac{25}{t^2}$

Use the power rule to find

$$f'(t) = 28t^3 - \frac{2}{\sqrt{t}} - \frac{50}{t^3}.$$

You could also write the derivative as

$$f'(t) = 28t^3 - \tfrac{1}{2}t^{-1/2} - 50t^{-3}.$$

b. $g(x) = \sin 5x + \tan 3x$

You'll need the chain rule to find the derivative of each term.

$$g'(x) = 5\cos 5x + 3\sec^2 3x$$

c. $y = \dfrac{e^x + \ln x}{5 + \sqrt{x}}$

Use the quotient rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^x + \ln x)'(5+\sqrt{x}) - (e^x + \ln x)(5+\sqrt{x})'}{(5+\sqrt{x})^2} \\ &= \frac{(e^x + 1/x)(5+\sqrt{x}) - (e^x + \ln x)\dfrac{1}{2\sqrt{x}}}{(5+\sqrt{x})^2} \end{aligned}$$

d. $f(x) = x \arctan x$. (Note that the inverse tangent function $\arctan x$ is often written $\tan^{-1} x$, but it does not equal $(\tan x)^{-1}$.)

Use the product rule.

$$\begin{aligned} f'(x) &= (x)' \arctan + x(\arctan x)' \\ &= \arctan x + \frac{x}{1+x^2} \end{aligned}$$

e. $f(\theta) = \theta^3 \cos\sqrt{\theta}$

Use the product rule and then the chain rule.

$$\begin{aligned} f'(\theta) &= (\theta^3)'(\cos\sqrt{\theta}) + (\theta^3)(\cos\sqrt{\theta})' \\ &= 3\theta^2(\cos\sqrt{\theta}) - \theta^3(\sin\sqrt{\theta})\frac{1}{2\sqrt{\theta}} \end{aligned}$$