Derivatives
Math 120 Calculus I
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Since we have a good understanding of limits, we can develop derivatives very quickly.

Recall that we defined the derivative $f'(x)$ of a function f at x to be the value of the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Sometimes the limit doesn't exist, and then we say that the function is not differentiable at x, but usually the limit does exist and, so, the function is differentiable.

Derivatives of linear functions. The graph of a linear function $f(x) = ax+b$ is a straight line with slope a. We expect that the derivative $f'(x)$ should be the constant slope a, and that's what we find it is when we apply the definition of derivative.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(a(x+h)+b)-(ax+b)}{h} \\
&= \lim_{h\to 0} \frac{ax+ah+b-ax-b}{h} \\
&= \lim_{h\to 0} \frac{ah}{h} \\
&= \lim_{h\to 0} a \\
&= a
\end{aligned}$$

Derivatives of constant functions. We've actually shown that the derivative of a constant function $f(x) = b$ is 0. That's because a constant function is a special case of a linear function where the coefficient of x is 0. It makes sense that the derivative of a constant is 0, since the slope of the horizontal straight line $y = b$ is 0.

Derivatives of powers of x. We'll directly compute the derivatives of a few powers of x like x^2, x^3, $1/x$, and $\sqrt{x}$. Note that these last two are actually powers of x even though we usually don't write them that way. The reciprocal of x is x raised to the power -1, that is, $1/x = x^{-1}$. Also, the square root of x is x raised to the power $\frac{1}{2}$, that is, $\sqrt{x} = x^{1/2}$. After computing these, we'll see that they all fit the pattern that says the derivative of x^n is nx^{n-1}. This pattern is called *the power rule* for derivatives.

First, we'll find the derivative of $f(x) = x^2$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x
\end{aligned}$$

Thus, the derivative of x^2 is $2x$.

Next, we'll compute the derivative of $f(x) = x^3$. At one point in the computation, we need to expand the cube of a binomial $(x+h)^3$, and that's something we could do by expanding $(x+h)(x+h)(x+h)$. The rule that gives the expansion of the general power $(x+h)^n$ is called the *binomial theorem*, and that's related to Pascal's triangle.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}$$

Thus, the derivative of x^3 is $3x^2$. Note that we didn't take the limit until we cancelled the h. That's always what happens.

Now, let's try to find the derivative of $f(x) = 1/x$. We start the same way as always

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1/(x+h) - (1/x)}{h}
\end{aligned}$$

At this point, put the numerator over a common denominator, then simplify the compound quotient before continuing.

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h} \\
&= \lim_{h\to 0} \frac{x-(x+h)}{(x+h)xh} \\
&= \lim_{h\to 0} \frac{-h}{(x+h)xh} \\
&= \lim_{h\to 0} \frac{-1}{(x+h)x} \\
&= \frac{-1}{x^2}
\end{aligned}$$

Thus, the derivative of $1/x$ is $-1/x^2$. This result fits the power rule mentioned above since we can rewrite it to say the derivative of x^{-1} is $-x^{-2}$.

It's a bit harder to compute the derivative of $\sqrt{x}$ since at one point we have to multiply the numerator and denominator of a quotient by a conjugate, but otherwise it's about the same difficulty. You might think that multiplying and dividing by a conjugate is a trick, but since we do it so often, you should think of it as a technique. Let $f(x) = \sqrt{x}$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Thus, the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$. Again, this fits the power rule, since it says the derivative of $x^{1/2}$ is $\frac{1}{2}x^{-1/2}$.

Here's a summary of the computations we've done so far.

$f(x)$	$f'(x)$
$ax+b$	a
x^2	$2x$
x^3	$3x^2$
$1/x$	$-1/x^2$
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$

Most of these results are special cases of the power rule which says that the derivative of x^n is nx^{n-1}.

Rules of differentiation. We could go on like this for each time we wanted the derivative of a new function, but there are better ways. There are several rules that apply in broad cases. In fact, we'll eventually find enough rules for differentiation that we won't need to go back to the definition in terms of limits. So, let's get started.

The sum rule. Let's begin with the rule for sums of functions. Many functions are sums of simpler functions. For example x^3+x^2 is the sum of two functions we've already differentiated. If we can discover the differentiation rule for sums, we'll be able to differentiate x^3+x^2 without going back to the definition of derivative.

Let f and g be two functions whose derivatives f' and g' we already know. Can we find the derivative $(f+g)'$ of their sum $f+g$? We'll need the definition of derivative to do that. When we apply the definition, we get

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}.$$

Now, the expression $(f+g)(x)$ means $f(x)+g(x)$, therefore, the expression $(f+g)(x+h)$ means $f(x+h)+g(x+h)$. We can continue as follows.

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0} \frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right) \\
&= \left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \right) \\
&\quad + \left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \right) \\
&= f'(x)+g'(x)
\end{aligned}$$

Thus, we have shown that the derivative $(f+g)'$ of the sum $f+g$ equals the sum $f'+g'$ of the derivatives.

This is a very useful rule. For instance, we can use it to conclude that the derivative of x^3+x^2 is $3x^2+2x$ because we already know the derivative of x^3 is $3x^2$ and the derivative of x^2 is $2x$.

The difference rule. If you just look back through the previous paragraph and change some plus signs to minus signs, you'll see that the derivative $(f-g)'$ of the difference $f-g$ equals the difference $f'-g'$ of the derivatives. So, for instance, the derivative of x^3-x^2 is $3x^2-2x$.

Constant multiple rule. On the same level of difficulty as addition and subtraction is multiplcation by constants. If f is a function, and c is a constant, then cf is the function whose value at x is $c\,f(x)$. We can easily find the derivative of cf in terms of the derivative of f.

$$\begin{aligned}
(cf)'(x) &= \lim_{h\to 0} \frac{(cf)(x+h)-(cf)(x)}{h} \\
&= \lim_{h\to 0} \frac{c\,f(x+h)-c\,f(x)}{h} \\
&= \lim_{h\to 0} \frac{c(f(x+h)-f(x))}{h} \\
&= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= c\,f'(x)
\end{aligned}$$

Thus, the derivative of a constant times a function is that constant times the derivative of the function, that is, $(cf)' = c\,f'$.

Derivatives of polynomials. With the help of the power rule, we can find the derivative of any polynomial. For example, the derivative of $10x^3 - 7x^2 + 5x - 8$ is $30x^2 - 14x + 5$. Finding derivatives of polynomials is so easy all you have to do is write down the answer, but here are the details so you can see that we're using all the rules we have so far. We'll use the abbreviated notation $(10x^3 - 7x^2 + 5x - 8)'$ for the derivative of $10x^3 - 7x^2 + 5x - 8$.

$(10x^3 - 7x^2 + 5x - 8)'$ equals, by the sum and difference rules,
$(10x^3)' - (7x^2)' + (5x)' - (8)'$ which equals, by the constant multiple rule,
$10(x^3)' - 7(x^2)' + 5(x)' - (8)'$ which equals, by the power rule and constant rule,
$10(3x^2) - 7(2x) + 5(1) - 0$, which simplifies to
$30x^2 - 14x + 5$, the answer.

The product, reciprocal, and quotient rules. These three rules are harder to prove, so we'll put off the proofs for a little bit. First we'll state them, then use them, and finally prove them. Let f and g be two differentiable functions.

$$\begin{aligned}
\text{Product rule:}\quad & (fg)' = f'g + fg' \\
\text{Reciprocal rule:}\quad & \left(\frac{1}{g}\right)' = \frac{-g'}{g^2} \\
\text{Quotient rule:}\quad & \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}
\end{aligned}$$

For our example to illustrate the use of the product rule, let's find the derivative of $x\sqrt{x}$. Here, $f(x) = x$ while $g(x) = \sqrt{x}$. Then

$$\begin{aligned}
(x\sqrt{x})' &= f'(x)g(x) + f(x)g'(x) \\
&= 1\sqrt{x} + x\,\frac{1}{2\sqrt{x}}
\end{aligned}$$

and this last expression simplifies to $\frac{3}{2}\sqrt{x}$. What's important to see in this example is how to use the product rule. In words, the product rule says that the derivative of the product fg of two functions equals the derivative f' of the first times the second g plus the first f times the derivative g' of the second.

Unlike sums and differences, the derivative of the product is *not* the product of the derivatives. That is to say, $(fg)'$ does *not* equal $f'g'$.

Next, let's have an example to illustrate the use of the reciprocal rule. Let's find the derivative of $\dfrac{1}{5x^3 - x + 2}$. This is the reciprocal of $5x^3 - x + 2$, and we know the derivative of $5x^3 - x + 2$ is $15x^2 - 1$. The reciprocal rule says

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}.$$

In this example $g(x) = 5x^3 - x + 2$ and $g'(x) = 15x^2 - 1$. Therefore,

$$\left(\frac{1}{5x^3 - x + 2}\right)' = \frac{-(15x^2 - 1)}{(5x^3 - x + 2)^2}.$$

Finally, let's have an example to illustrate the use of the quotient rule. Let's find the derivative of $\dfrac{6x^3 - 8x^2}{x^2 + 2x + 8}$. The quotient rule says

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

In this example, $f(x) = 6x^3 - 8x^2$ and $g(x) = x^2 + 2x + 8$. We know the derivatives of f and g. They are $f'(x) = 18x^2 - 16x$ and $g'(x) = 2x + 2$. Therefore,

$$\left(\frac{6x^3 - 8x^2}{x^2 + 2x + 8}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$
$$= \frac{(18x^2 - 16x)(x^2 + 2x + 8) - (6x^3 - 8x^2)(2x + 2)}{(x^2 + 2x + 8)^2}.$$

Of course, we can simplify our answer, but for the purposes of this example there is no need to do so.

The derivative of the quotient is complicated. You can remember the rule best in words. It says that the derivative of a quotient f/g is the derivative f' of the numerator times the denominator g plus the numerator f times the derivative g' of the denominator, all divided by the square g^2 of the denominator.

You'll have to use the product, reciprocal, and quotient rules several times before you remember them well. You can actually do without the reciprocal rule, since it's a special case of the quotient rule, but it comes up often enough so that it's worth while to memorize it.

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