Proofs of the Product, Reciprocal, and Quotient Rules
Math 120 Calculus I
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So far, we've defined derivatives in terms of limits

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h};$$

found derivatives of several functions; used and proved several rules including the constant rule, sum rule, difference rule, and constant multiple rule; and used the product, reciprocal, and quotient rules. Next, we'll prove those last three rules.

After that, we still have to prove the power rule in general, there's the chain rule, and derivatives of trig functions. But then we'll be able to differentiate just about any function we can write down.

The product, reciprocal, and quotient rules. For the statement of these three rules, let f and g be two differentiable functions. Then

$$\text{Product rule:} \quad (fg)' = f'g + fg'$$
$$\text{Reciprocal rule:} \quad \left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$
$$\text{Quotient rule:} \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

A more complete statement of the product rule would assume that f and g are differentiable at x and conlcude that fg is differentiable at x with the derivative $(fg)'(x)$ equal to $f'(x)g(x) + f(x)g'(x)$. Likewise, the reciprocal and quotient rules could be stated more completely.

A proof of the product rule. This proof is not simple like the proofs of the sum and difference rules. By the definition of derivative,

$$(fg)'(x) = \lim_{h \to 0} \frac{(fg)(x+h) - (fg)(x)}{h}.$$

Now, the expression $(fg)(x)$ means $f(x)g(x)$, therefore, the expression $(fg)(x+h)$ means $f(x+h)g(x+h)$. So we have

$$(fg)'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

Now we have a problem. The numerator

$$f(x+h)g(x+h) - f(x)g(x)$$

is a difference that involves x changing to $x+h$ for both f and g. We need to isolate the change for f from the change for g.

Let's look at this geometrically. Suppose $f(x)$ is the length of an elongating horizontal line segment and $g(x)$ is the length of an elongating vertical line segment. Then $f(x)g(x)$ is the area of an enlarging rectangle. $f(x+h)$ is the length of the horizontal line segment at a later time $x+h$, and $g(x+h)$ is the length of the vertical line segment at that later time, so $f(x+h)g(x+h)$ is the area of a larger rectangle.

The difference $f(x+h)g(x+h) - f(x)g(x)$ is the difference of these two rectangles. It's made out of the three components labelled Area A, Area B, and Area C in the figure.

$g(x+h) - g(x)$	Area B	Area C
$g(x)$		Area A
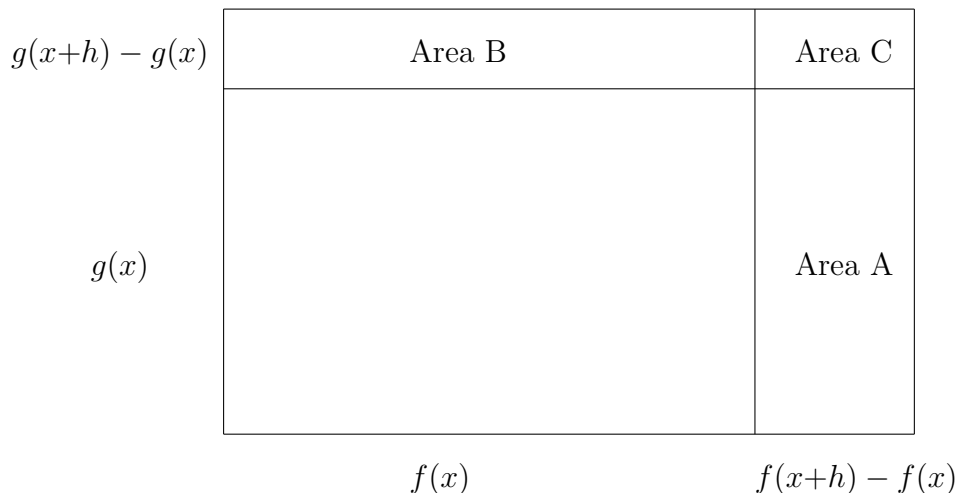
$$f(x) \qquad\qquad\qquad f(x+h) - f(x)$$

So we can rewrite the numerator $f(x+h)g(x+h) - f(x)g(x)$ as a sum of three terms

$$
\begin{array}{ccccc}
& \text{Area A} & + & \text{Area B} & + & \text{Area C} \\
= & (f(x+h) - f(x))g(x) & + & f(x)(g(x+h) - g(x)) & + & (f(x+h) - f(x))(g(x+h) - g(x))
\end{array}
$$

corresponding to those three areas. Now we can evaluate the derivative of the product.

$$
\begin{aligned}
& (fg)'(x) \\
= & \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
= & \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x) + f(x)(g(x+h) - g(x)) + (f(x+h) - f(x))(g(x+h) - g(x))}{h} \\
= & \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h) - g(x))}{h} + \lim_{h\to 0} \frac{(f(x+h) - f(x))(g(x+h) - g(x))}{h} \\
= & \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} (g(x+h) - g(x)) \\
= & f'(x)g(x) + f(x)g'(x) + f'(x)\,0
\end{aligned}
$$

Only that last 0 needs explanation. Why is $\lim_{h\to 0}(g(x+h) - g(x)) = 0$? But, surely as $h \to 0$, doesn't $g(x+h) \to g(x)$? Well, yes, if g is continuous at x. But do we know g is continuous? We will in a moment when we prove that if g is differentiable at x then g is

continuous at x. Remember that we're assuming for this proof of the product rule that both f and g are continuous at x. So, after the next proof, we'll be justified in finishing this one to conclude

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Differentiability implies continuity. We'll prove now that if a function g is differentiable at x, then g has to be continuous at x. This proof is straightforward, that is to say, there aren't any tricks in it. Now, since g is differentiable at x, therefore the limit

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

exists. But the denominator is approaching 0, so the only way the limit can exist is if the numerator also approaches 0, that is,

$$\lim_{h \to 0} (g(x+h) - g(x)) = 0.$$

That implies

$$\lim_{h \to 0} g(x+h) = g(x),$$

which says g is continuous at x.

A proof of the reciprocal rule. Now that we've proved the product rule, it's time to go on to the next rule, the reciprocal rule. We need to prove that

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{(g(x))^2}.$$

Our assumptions include that g is differentiable at x and that $g(x) \neq 0$.

The argument is pretty much the same as the computation we used to show the derivative of $1/x$ was $-1/x^2$. We'll use the definition of derivative.

$$\left(\frac{1}{g}\right)'(x)$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{g}\right)(x+h) - \left(\frac{1}{g}\right)(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{g(x+h)}\right) - \left(\frac{1}{g(x)}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{g(x) - g(x+h)}{g(x+h)g(x)}\right)}{h}$$

$$= \lim_{h \to 0} \frac{g(x) - g(x+h)}{g(x+h)g(x)h}$$

$$= \left(\lim_{h \to 0} \frac{g(x) - g(x+h)}{h}\right) \left(\lim_{h \to 0} \frac{1}{g(x+h)}\right) \frac{1}{g(x)}$$

Note that the first limit is $-g'(x)$ since g is differentiable at x. Also note that the second limit is $1/g(x)$ since g is continuous at x (since it's differentiable there) and $g(x) \neq 0$. Therefore, we conclude

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{(g(x))^2}.$$

A proof of the quotient rule. This will be easy since the quotient f/g is just the product of f and $1/g$. So, to prove the quotient rule, we'll just use the product and reciprocal rules. We don't even have to use the definition of derivative.

First, treat the quotient f/g as a product of f and the reciprocal of g.

$$\left(\frac{f}{g}\right)' = \left(f \cdot \left(\frac{1}{g}\right)\right)'$$

Next, apply the product rule.

$$\left(f \cdot \left(\frac{1}{g}\right)\right)' = f' \cdot \left(\frac{1}{g}\right) + f \cdot \left(\frac{1}{g}\right)'$$

and apply the reciprocal rule to find $(1/g)'$ to see that equals

$$f' \cdot \left(\frac{1}{g}\right) + f \cdot \left(\frac{-g'}{g^2}\right)$$

Finish by writing the expression with a common denominator of g^2 to get

$$\frac{f'g - fg'}{g^2}$$

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