Math 128, Modern Geometry

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Due Today. Exercises from chapter 4: exercises 1, 2, 5.

Today. Begin Möbius geometry.

The definition of Möbius geometry. We'll describe this geometry via a transformation group on a set, then investigate its invariants. The underlying space of Möbius geometry is $\mathbf{C}^+$, the complex plane with one element, the point at ∞, added. A transformation $T : \mathbf{C}^+ \to \mathbf{C}^+$ of this geometry is of the form

$$T(z) = \frac{az+b}{cz+d}$$

where a, b, c, and d are complex constants such that $ad - bc \neq 0$. You probably recognize $ad - bc$ as the determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. There's a way of analyzing all the geometry we're doing in terms of matrices and linear algebra, but we won't be looking at that. Such a transformation is called a *Möbius transformation* or a *fractional linear transformation.*

These Möbius transformations do in fact form a transformation group G becase they have the three properties required to be a transformation group.

First, the identity transformation I is included in G. That's the transformation I defined by $I(z) = z = \frac{1z+0}{0z+1}$ is a Möbius transformation.

Second, for each transformation $T \in G$, there is an inverse transformation $T^{-1} \in G$ because

$$w = T(z) = \frac{az+b}{cz+d}$$

if and only if

$$z = T^{-1}(w) = \frac{dw-b}{-cw+a}.$$

Third, the composition $T \circ U$ of two Möbius transformations is another one. (See the text for details.)

Geometric interpretation of Möbius transformations. These Möbius transformations include several of the transformations we've looked at so far. All translations $T(z) = z + b = \frac{1z+b}{0z+1}$ are Möbius transformations. Also, all rotations about the origin $T(z) = e^{i\theta}z$ and scalings fixing the origin $T(z) = rz$ (with $r > 0$) are Möbius transformations since these are both of the form $T(z) = \frac{az+0}{0z+1}$. And, since Möbius transformations are closed under composition, that implies rotations and scalings about any point are Möbius transformations. Furthermore, reciprocation $T(z) = \frac{1}{z} = \frac{0z+1}{1z+0}$ is a Möbius transformation.

But reflections are not Möbius transformations. In particular, complex conjugation is not a Möbius transformation.

On the other hand, every Möbius transformation can be decribed as a composition of translations, rotation, scalings, and reciprocation. When $c \neq 0$,

$$T(z) = \frac{az+b}{cz+d} = \frac{1}{c} - \frac{ad-bc}{c^2}\left(z + \frac{d}{c}\right)^{-1},$$

which illustrates T as a composition of these various operations. Likewise, when $c = 0$,

$$T(z) = \frac{az+b}{d} = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)$$

shows T is also a composition of these various operations.