CLARK UNIVERSITY 1887

Fifth assignment
Math 217 Probability and Statistics
Prof. D. Joyce, Fall 2014

1. A box has numbers from 1 to 10. A number is drawn at random. Let X_1 be the number drawn. This number is replaced, and the ten numbers mixed. A second number X_2 is drawn. Find the distributions of X_1 and X_2. Are X_1 and X_2 independent? Answer the same questions if the first number is not replaced before the second is drawn.

2. A die is rolled twice. Let X denote the sum of the two numbers that turn up, and Y the difference of the numbers (specifically, the number on the first roll minus the number on the second). Show that $E(XY) = E(X)E(Y)$. Are X and Y independent?

3. Exactly one of six similar keys opens a certain door. If you try the keys, one after another, what is the expected number of keys that you will have to try before success?

4. Show that, if X and Y are random variables taking on only two values each, and if $E(XY) = E(X)E(Y)$, then X and Y are independent.

Suggestion: Let X take values a and b, and let Y take values c and d. Let the joint probability for (X, Y) be as given in this table

	c	d	
a	p	q	$p+q$
b	r	s	$r+s$
	$p+r$	$q+s$	

where $p + q + r + s = 1$.

5. A royal family has children until it has a boy or until it has three children, whichever comes first. Assume that there are no twins, and each child is a boy with probability $\frac{1}{2}$. Find the expected number of boys in this royal family and the expected number of girls.

6. A box contains two gold balls and three silver balls. You are allowed to choose successively balls from the box at random. You win \$1 each time you draw a gold ball and lose \$1 each time you draw a silver ball. After a draw, the ball is not replaced. Show that, if you draw until you are ahead by \$1 or until there are no more gold balls, this is a favorable game.

7. Gerolamo Cardano in his book, *The Gambling Scholar,* written in the early 1500s, considers the following carnival game. There are six dice. Each of the dice has five blank sides. The sixth side has a number between 1 and 6—a different number on each die. The six dice are rolled and the player wins a prize depending on the total of the numbers which turn up.

Find, as Cardano did, the expected total without finding its distribution.

Large prizes were given for large totals with a modest fee to play the game. Explain why this could be done.

8. A number is chosen at random from the set $S = \{-1, 0, 1\}$. Let X be the number chosen. Find the mean $\mu = E(X)$, variance $\sigma^2 = \mathrm{Var}(X)$, and standard deviation σ of X.

9. A random variable X takes the four values 0, 1, 2, and 4 with probabilities $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{6}$, respectively. Find the mean, variance, and standard deviation of X.

10. A coin is tossed three times. Let X be the number of heads that turn up. Find the mean, variance, and standard deviation of X.

11. A number is chosen at random from the integers 1, 2, 3,…, n. Let X be the number chosen. Show that $E(X) = (n+1)/2$ and $\mathrm{Var}(X) = \frac{1}{12}(n-1)(n+1)$. *Hint*: The following identity will be useful:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} .$$

12. Let X be a random variable with $\mu = E(X)$ and $\sigma^2 = \mathrm{Var}(X)$. Define $X^* = (X - \mu)/\sigma$. The random variable X^* is called the *standardized random variable* associated with X. Show that this standardized random variable has expected value 0 and variance 1.

Math 217 Home Page at
`http://math.clarku.edu/~djoyce/ma217/`