Bayes' formula
Math 217 Probability and Statistics
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Bayes' formula for inverting conditional probabilities. Frequently, we want to know $P(F|E)$, but we know $P(E|F)$ instead. How do you invert these conditional probabilities? Bayes, long before the formal foundations of probability were invented, that is, before the concept of sample space, figured out a way.

Actually, it's easy to see what Bayes' formula has to be. Since $P(F \cap E)$ equals both $P(F|E)\,P(E)$ and $P(E|F)\,P(F)$, therefore

$$\boxed{P(F|E) = \frac{P(E|F)\,P(F)}{P(E)}}$$

When applying this formula, it is often the case that $P(E)$ has to be computed from conditional probabilities. Since $P(E) = P(E \cap F) + P(E \cap F^c) = P(E|F)\,P(F) + P(E|F^c)\,P(F^c)$, therefore

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E|F)\,P(F) + P(E|F^c)\,P(F^c)}.$$

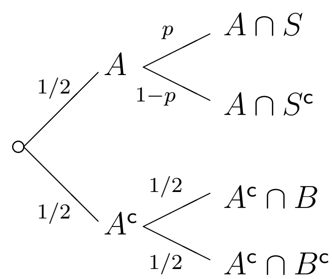
An example. In order to encourage subjects to be honest in a psychological experiment, before answering a particularly embarrassing question, such as "Do you sing in the shower?" they are told to flip two coins. If the first coin is heads, they should answer the question honestly Yes or No, but if the first coin is tails, then they should answer the question yes if the second coin is heads but No if that coin is tails. The idea behind this kind of experiment is to encourage truthful responses half the time since the subject knows that his or her responses can't be used to conclude anything about the subject.

Suppose this experiment is carried out with 100 subjects, and 70 of them answer Yes and 30 of them No. What is a good estimate of the probability that they sing in the shower?

Let's label the events with letters to be able to refer to them more easily. Let A be the event that the first coin flip is Heads, so that A^c is the event that the first coin flip is Tails. Let B be the event that the second coin is Heads. Let S be the event that the subject sings in the shower, and let T be the event that the subjects says he/she sings in the shower. Finally, let p denote the unknown probability $P(A)$ that the subject sings in the shower.

This can be seen most easily with a tree diagram. The first branching of the tree represents the first coin flip. One branch leads to A with probability $\frac{1}{2}$, the other to A^c, also with probability $\frac{1}{2}$.

At A, branch two ways depending on whether the subject sings in the shower. One way leads to the node $A \cap S$, the other leads to the node $A \cap S^c$. We'll label the branches with conditional probabilities so that the one that leads from A to $A \cap S$ is labelled p, and the other branch is labelled $1-p$.



At A^c, branch two ways depending on the second coin flip. One branch, labelled $\frac{1}{2}$ leads to the node $A^c \cap B$, the other, also labelled $\frac{1}{2}$ leads to the node $A^c \cap B^c$.

Now, T, the event that the subject says he/she sings in the shower, is the union of two of the leaves of this tree, namely, $A \cap S$ and $A^c \cap B$. So, the probability of T is the sum of the probabilities of these two leaves. Algebraically, that's the equation

$$\begin{aligned} P(T) &= P((A \cap S) \cup (A^c \cap B)) \\ &= P(A \cap S) + P(A^c \cap B) \\ &= P(A)\,P(S|A) + P(A^c)\,P(B|A^c) \\ &= \tfrac{1}{2}p + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{2}p + \tfrac{1}{4} \end{aligned}$$

where we read the conditional probabilities off the tree diagram. (We used the multiplicative principle for conditional probabilities.)

Since a good estimate of $P(T)$ is $\frac{70}{100}$, we can find a good estimate of p by solving the equation $\frac{70}{100} = \frac{1}{2}p + \frac{1}{4}$, and that gives $p = \frac{90}{100}$.

Let's go on to another question. Given a subject said he/she does sing in the shower, what's the probability he/she actually sings in the shower? In other words, what is the conditional probability $P(S|T)$?

The reverse probability $P(T|S)$ is easy to find by following the tree diagram. If a subject does sing in the shower, then with probability $\frac{1}{2}$, the first coin is Heads, so the subject says truthfully he/she sings in the shower, but with probability $\frac{1}{2}$ the first coins is heads, and with equal probability says he/she does or doesn't sing. Thus, $P(T|S) = \frac{3}{4}$.

But we don't want the reverse probabilities, so we can use Bayes' formula. It says

$$P(S|T) = \frac{P(T|S)\,P(S)}{P(T)} = \frac{\frac{3}{4} \cdot \frac{90}{100}}{\frac{70}{100}} = \frac{27}{28}.$$

People v. Collins. (From Grinstead and Snell.)

In this case a purse was snatched from an elderly person in a Los Angeles suburb. A couple seen running from the scene were described as a black man with a beard and a mustache and a blond girl with hair in a ponytail. Witnesses said they drove off in a partly yellow car. Malcolm and Janet Collins were arrested. He was black and though clean shaven when arrested had evidence of recently having had a beard and a mustache. She was blond and usually wore her hair in a ponytail. They drove a partly yellow Lincoln. The prosecution called a professor of mathematics as a witness who suggested that a conservative set of probabilities for the characteristics noted by the witnesses would be as shown in the following table.

man with mustache	1/4
girl with blond hair	1/3
girl with ponytail	1/10
black man with beard	1/10
interracial couple in a car	1/1000
partly yellow car	1/10

Table 1: Collins case probabilities.

The prosecution then argued that the probability that all of these characteristics are met by a randomly chosen couple is the product of the probabilities or 1/12,000,000, which is very small. He claimed this was proof beyond a reasonable doubt that the defendants were guilty. The jury agreed and handed down a verdict of guilty of second-degree robbery.

If you were the lawyer for the Collins couple how would you have countered the above argument?

Of course there are many possible counterarguments, but let's see if we can come up with some that have to do with the probabilistic concepts currently under consideration.

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