Combinatorics
Math 217 Probability and Statistics
Prof. D. Joyce, Fall 2014

We'll need to count some things in probability. The mathematics of counting is called combinatorics. Actually, combinatorics doesn't really involve counting, but finding out how many things there are *without* counting. We won't need much except the most introductory topics from the field.

We'll look at additive and multiplicative principles, permutations, combinations, binomial coefficients and Pascal's triangle, and multinomial coefficients.

Additive principles. The basic additive principle you've known since you first learned addition. If you have so many of these and those, and none of these are those, then you can find out how many of these and those you have altogether by adding them.

Let's introduce some notation and terminology. If S is a set, then we'll call the number of elements it has its *cardinality*, and we'll denote it's cardinality with absolute value signs. Thus, if $S = \{a, b, c\}$, then the cardinality of S is 3, written $|S| = 3$.

The basic additive principle says that if S and T are disjoint sets, that is to say, their intersection is empty, $S \cap T = \emptyset$, then the cardinality of their union is the sum of their cardinalities. $|S \cup T| = |S| + |T|$.

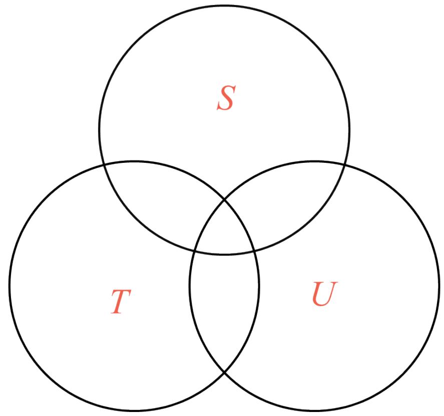
But what if S and T aren't disjoint? You can't use that formula since you would count elements in the intersection twice. That's easy to fix. Just subtract the cardinality of their intersection.

The principle of inclusion and exclusion. That's what that last observation is called. For any two sets S and T, it is the case that $|S \cup T| = |S| + |T| - |S \cap T|$. You include S and T, then exclude their intersection.

This principle works for probability, too. Here's an example. A standard deck of cards has 52 cards with 13 cards in each suit. If you draw one at random you have the same probability of drawing any one of the 52 cards. This is uniform probability where each outcome has a probability of $\frac{1}{52}$. Since there are 13 spades, the probability of drawing a spade is $P(S) = \frac{13}{52} = \frac{1}{4}$. Also, since there are 4 tens, the probability of drawing a ten is $P(T) = \frac{4}{52} = \frac{1}{13}$. Now, what's the probability of drawing a space or a ten. It's not $P(S) + P(T)$ since that counts the ten of spades twice. It's $P(S \cup T) = P(S) + P(T) - P(S \cap T) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$.

The principle of inclusion and exclusion extends to more than two sets. Suppose you have three sets S, T, and U. What's the cardinality of their union $S \cup T \cup U$? You can start by summing $|S| + |T| + |U|$, but, as before that counts their intersections twice. So subtract the three intersections, $-|S \cap T| - |S \cap U| - |T \cap U|$. But now you've subtracted the triple intersection $S \cap T \cap U$ three times. It was added three times in the initial sum, so now it's not counted at all. That's easy to fix. Just add the cardinality of the triple intersection. Thus

$$|S \cup T \cup U| = |S| + |T| + |U| - |S \cap T| - |S \cap U| - |T \cap U| + |S \cap T \cap U|.$$



Summary: to find the cardinality of a triple union, add the cardinality of each set, subtract the cardinalities of the double intersections, and add the cardinality of the triple intersection.

That last formula generalizes. If you have n sets, you keep adding and subtraction multiple intersections until you add or subtract the intersection of all n sets. Here's a formula that summarizes that.

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le n} |S_{i_1}\cap\cdots\cap S_{i_k}|$$

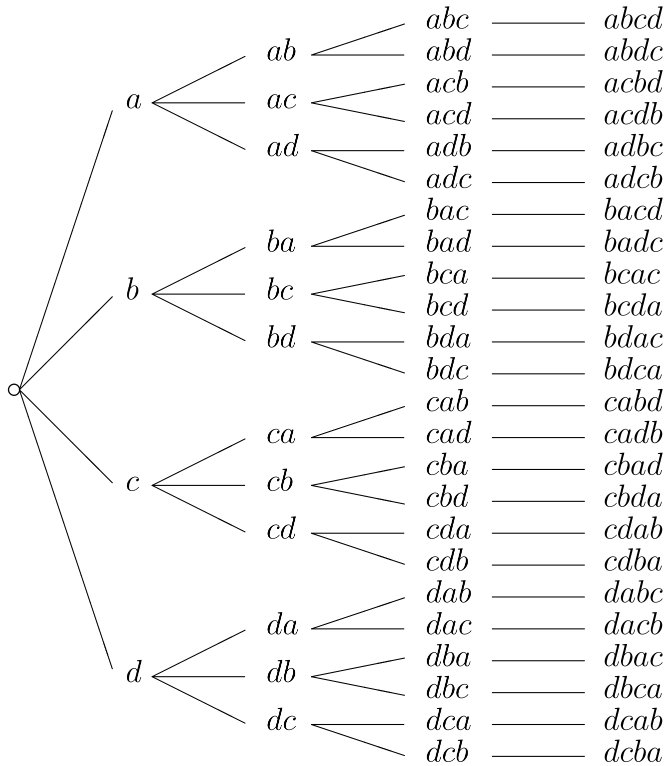
The multiplicative principle. The basic multiplicative principle says that if you have m choices, and for each choice you have n second choices (and all the second choices for one of the first choices differs from all the second choices for any of the other first choices), then altogether you have mn choices.

One situation in which this occurs is when you take the cartesian product of two sets S and T. The cartesian product $S\times T$ consists of all ordered pairs (s,t) where $s\in S$ and $t\in T$. Then $|S\times T| = |S|\,|T|$. In most of our applications, however, what the second choices are depend on the first choice you make, so we're not looking at just cartesian products of sets.

The multiplicative principle also works when there are more than two stages. For example, if there are three stages with m choices at the first stage, n at the second, and p at the third, then there are mnp altogether.

Permutations. One of the primary applications of the multiplicative principle is counting permutations. Suppose we want to count all the ways you can rearrange the letters in ROFL. There are a lot of them such as FROL, OLFR, etc. These rearrangements are called *permutations*. When choosing a permutation of ROFL, you have 4 choices for the first letter, 3 remaining choices for the second (since we can't choose the first letter again), 2 remaining choices for the third, and then the fourth is forced on us. Thus there are $4\cdot 3\cdot 2\cdot 1 = 24$ choices altogether.

We'll be using tree diagrams a lot in this course, and here's a good place to introduce them. When choosing a permutation of the four letters *abcd* there are four stages. The first stage chooses one of the four letters to go first. That gives us our first branching of the tree at the left. After we've taken that branch, we'll be at one of the four *nodes* or *states* labelled a, b, c, or d. At this second stage, we choose a second letter that can't be the same as the first. In each case we have three choices this time, so we'll take one of the three branches to get to a state labelled by two letters. At the third stage, we've got two choices, so for each state there are two branches leading to a state labelled with three letters. At this state the last letter is determined, so there's only one branch to a *leaf* of the tree.



With a tree diagram like this you can visually see how the multiplication principle increases the number of states at each stage by a factor equal to the number of braches at each state.

In general, there will be n factorial permutations of n things where n factorial, written $n!$, is the product of the integers from 1 though n. It's also useful to define 0! to be 1 in order to make things easier to say when factorials are involved.

For probability, that's about all we need to know about permutations although sometimes variants come up. Here's one. Say you want to count the permutations of ROFLCOPTER. The two R's can't be distinguished, so exchanging them shouldn't count as a different permutation. If we said that there are 10! permutations, we would be doubly counting them because the R's aren't distinguishable. There are two O's, too, so the actual number of distinguishable permutations is 10! divided by 4.

Likewise, when you're counting distinguishable permutations of MISSISSIPPI, you'll need to divide 11! by 4! because of the repeated I's, 4! because of the repeated S's, and 2! because of the repeated P's.

Sterling's approximation for factorials. Sometimes you'll need to compute factorials of large numbers. Sterling's approximation helps. The factorial function $n!$ grows very fast with n. James Sterling (1692–1770) this approximation for factorials:

$$\boxed{n! \approx n^n e^{-n} \sqrt{2\pi n}}$$

This approximation is fairly good even for numbers as small as 10 where the approximation has an error of less than 1%. It's accuracy increases with n.

n	$n!$	approx	ratio
1	1	0.922137	1.084
2	2	1.91900	1.042
3	6	5.83621	1.028
4	24	23.5062	1.021
5	120	118.019	1.016
6	720	710.078	1.014
7	5040	4980.40	1.012
8	40320	39902.4	1.011
9	362880	359536	1.0093
10	3628800	3598690	1.0084
11	39916800	39615600	1.0076
12	479001600	475687000	1.0070

k-permutations. The permutations discussed above are full permutations of all n items. Sometimes we don't want full permutations of a set of n elements, but just partial permutations. If $k \leq n$, a *k-permutation* is an ordered listing of just k elements of a set of n elements. For instance, the 3-permutations of *abcd* are these

$$\begin{array}{cccc} abc & bac & cab & dab \\ abd & bad & cad & dac \\ acb & bca & cba & dba \\ acd & bcd & cbd & dbc \\ adb & bda & cda & dca \\ adc & bdc & cdb & dcb \end{array}$$

while the 2-permutations are these

$$\begin{array}{cccc} ab & ba & ca & da \\ ac & bc & cb & db \\ ad & bd & cd & dc \end{array}$$

We can determine how many k-permutations of a set of n elements there are using the multiplicative principle. In the first stage, choose one of the n elements to go first. In the second stage, there are $n-1$ remaining elements, and choose one of them to go second. At the next stage, chose one of the remaining $n-2$ elements to go next. And so forth until the kth stage, when there are $n-k+1$ remaining elements. Thus, the number of k-permutations of a set of n elements is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

There is no particular standard notation for the number of k-permutations of a set of n elements, but you'll see it denoted $(n)_k$, nPk, P_k^n, and various other things. We'll just use $n!/(n-k)!$ as we won't be using k-permutations very much.

We'll continue next time with a discussion of binomial coefficients, which count combinations, the binomial theorem, Pascal's triangle, and multinomial coefficients.

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