Assignment 2 answers
Math 105 History of Mathematics
Prof. D. Joyce, Spring 2015

Exercise 17. Convert the fractions 7/5, 13/15, 11/24, and 33/50 to sexagesimal notation.

It's your decision as to how close to the Babylonian method you perform the operations. To be pretty authentic, to find 7/5 you would find the reciprocal of 5, which is 0;12, and multiply that by 7. Since 7 times 12 is 60 plus 24, you'll find that $7/5 = 1;24$. (Alternatively, 7/5 is 1 plus 2/5, and since 1/5 is $0;12$, therefore 2/5 is $0;24$.)

For 13/15, since 1/15 is $0;04$, therefore $13/15 = 0;52$.

For 11/24, since 1/24 is half of 1/12, and 1/12 is $0;05$, and so 1/24 is $0;02,30$, therefore 11/24 is 11 times $0;02,30$, which is $0;27,30$.

Likewise, to find 1/50, note 1/5 is $0;12$, so $1/25 = 0;02,24$, and $1/50 = 0;01,12$, a value you can find on the reciprocal table. Therefore 33/50 is 33 times $0;01,12$ which, through a bit of computation is $0;39,36$.

Exercise 18. Convert the sexagesimal fractions 0;22,30, 0;08,06, 0;04,10, and 0;05,33,20 to ordinary fractions in lowest terms.

For the first one, note that 0;22,30 is $\frac{22}{60}$ plus $\frac{30}{60^2}$, and that equals $\frac{22\cdot60+30}{60^2} = \frac{1350}{60^2} = \frac{3}{8}$.

$0;08,06 = \frac{8}{60} + \frac{6}{60^2} = \frac{81}{600} = \frac{27}{200}$.

$0;04,10 = \frac{4}{60} + \frac{10}{60^2} = \frac{25}{360} = \frac{5}{72}$.

$0;05,33,20 = \frac{5}{60} + \frac{33}{60^2} + \frac{20}{60^3} = \frac{5}{60} + \frac{100}{3\cdot60^2} = \frac{5}{54}$.

Exercise 19. Find the reciprocals in base 60 of 18, 27, 32, 54, and 64 (=1,04). What is the condition on the integer n which insures that its reciprocal is a finite sexagesimal fraction?

The Babylonians would have found these reciprocals from their tables that they built up starting with 1, 2, 3,... . This problem is mainly for you to become familiar with base 60.

So, how, using whatever methods you have, can you find the reciprocal of 18? Here's one way. Now, 1/18 is 1/60 of 60/18, which simplifies to 10/3, which is 3 plus 1/3, and 1/3 is 20/60. So 1/18 equals 3/60 plus $20/60^2$, or in the notation we're using, 0;3,20. Since the Babylonians didn't use a decimal point, they would have written it in cuneiform corresponding to 3,20. Using this method you can find reciprocals of the other numbers.

An easier way, and perhaps closer to the way the Babylonians did it, was to build up the table. Since 18 is twice 9, and 9 is three times 3, the reciprocal of 18 can be found in three steps. The reciprocal of 3 is 0;20, which without the decimal point is 20. One third of 20 is 6 $\frac{2}{3}$, that is ;06,40. Half of that is ;03,20

3	20
9	6, 40
18	3, 20

The other reciprocals can be likewise found by dividing by 2 and 3. Dividing the line 9 — 6:24 by 3 gives a line for 27. Halving that gives one for 54.

27	3, 13, 20
54	1, 06, 40

Repeated halving from $\frac{1}{2}$ will give reciprocals for 32 and 64:

2	30
4	15
8	7, 30
16	3, 45
32	1, 52, 30
64	56, 15

Now, what is the condition on n which insures that its reciprocal is a finite sexagesimal fraction? The same question can be asked about decimals in base ten, and you probably already know the answer to that: for any number n whose only prime factors are 2 and 5 it is the case that $1/n$ has a finite decimal representation. The analogous condition in base 60 uses 2, 3, and 5. So if n can be built out of multiplying 2, 3, and 5 only, then $1/n$ has a finite sexagesimal representation.

Exercise 20 In the Babylonian system, multiply 25 by 1,04 and 18 by 1,21 Divide 50 by 18 and 1,21 by 32 (using reciprocals). Use our standard multiplication algorithm modified for base 60.

The following computation does the first product.

	25	
1,	04	
	20	which is 5 times 4
1,	20	which is 20 times 4
25,		which is 1 times 25
26,	40	

	18	
1,	21	
	18	$1 \cdot 18$
2,	40	$20 \cdot 8$
3,	20	$20 \cdot 10$
18,		$1 \cdot 18$
24,	18	

The reciprocal of 18 is 0;03,20, and the product of 50 and 0;03,20 is 2;46,40.

The reciprocal of 32 is 0;01,52,30, and the product of 0;01,22,30 and 1,21 is 2;31,52,30.

Exercise 24. Convert the Babylonian approximation 1;24,51,10 of the square root of 2 to decimals and determine the accuracy of the approcimation.

This is most easily computed from right to left:

$$((10/60 + 51)/60 + 24)/60 + 1 = 1.414212963$$

which is pretty close to 1.414213562. Six digits of accuracy, and only off by 1 in the next digit.

Exercise 25. Use the assumed Babylonian square root algorithm of the text to show that the square root of 3 is about 1;45 by beginning with the value 2. Find a three-sexagesimal-place approximation to the reciprocal of 1;45 and use it to calculate a three-sexagesimal-place approximation to the square root of 3.

The method says starting with an approximate root a of n, a better approximation would be $a + b/(2a)$ where $b = n - a^2$. If we use a little modern algebra, we can simplify the expression for the better approximation to simply $\dfrac{a + n/a}{2}$, that is, the average of a and n/a.

With $n = 3$, and $a = 2$, we want the average of 2 and $3/2$, which is 1 and $3/4$, that is, 1;45.

Now, with n still 3, but the better approximation 1;45 to its square root, apply the method again to get a still better approximation. We want the average of 1;45 and 3 multiplied by the reciprocal of 1;45. Now 1;45 is $7/4$, so we want $4/7$ in sexagesimal. There are various ways to find that in sexagesimal. If you're willing to use a modern calculator, punch in $4/7$. Multiply by 60 and you see you get 34 plus a fraction. Subtract the 34 and multiply by 60 again. You get 17 plus a fraction. Subtract the 17 and multiply by 60 again. You get just under 9. Thus, the reciprocal is about ;34,17,09. Multiply that by $n = 3$ to get 1;42,51,27. Now average that with 1;45. Add the two together to get 3;27,51,27, then halve to get 1;43,55,43,30.

Decimally, that's 1.73214. As $\sqrt{3} = 1.732050808$ to 10 significant digits, this ancient Babylonian approximation was accurate to 4 significant digits.

Incidentally, this algorithm works very well, each iteration doubles the number of digits of accuracy. One more iteration would have given the Babylonians 8 digits of accuracy.

Exercise 28. Solve the problem from the Old Babylonian tablet BM 13901: The sum of the areas of two squares is 1525. The side of the second square is $2/3$ that of the first plus 5. find the sides of each square. (You may use modern methods to find the solution.)

If the sides of the two squares are x and y, then we have two equations

$$\begin{aligned} x^2 + y^2 &= 1525 \\ y &= \tfrac{2}{3}x + 5 \end{aligned}$$

Eliminating y gives one equation in x that simplifies to $\frac{13}{9}x^2 + \frac{20}{3}x - 1500 = 0$. You can solve this by standard methods to get $x = 30$. (The negative solution is not feasible.) So the two squares have sides 30 and 25.

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