Selected answers from assignment 7
Math 105 History of Mathematics
Prof. D. Joyce

The assignment Part II. Exercises from the text. Page 319, exercises 24 and 36, and on page 359, exercises 1 and 17.

Exercise 24. Find a pair of amicable numbers different from those in the text. (*Hint*: Try the case $n = 7$ of ibn Qurra's theorem.)

Ibn Qurra's theorem says that if $p_{n-1} = 3 \cdot 2^{n-1} - 1$, $p_n = 3 \cdot 2^n - 1$, and $q_n = 9 \cdot 2^{2n-1} - 1$ are three prime numbers, then $a = 2^n p_{n-1} p_n$ and $b = 2^n q_n$ are a pair of amicable numbers.

With $n = 7$, $p_6 = 3 \cdot 2^6 - 1 = 191$, $p_7 = 3 \cdot 2^7 - 1 = 383$, and $q_n = 9 \cdot 2^{13} - 1 = 73727$. It takes time to verify that these are all prime, especially 73,727. They are. Therefore $a = 2^7 p_6 p_7 = 9363684$ and $b = 2^7 q_7 = 9437056$ are amicable numbers.

Exercise 36. [Part 1.] Al-Biruni devised a method for determining the radius r of the earch by sighting the horizon from the top of a mountain of known height h. That is, al0Biruni assumed that one could mesaure α the angle of depression from the horizontal at which one sights the apparent horizon (see the figure in the text). Show that r is determined by the formula

$$r = \frac{h \cos \alpha}{1 - \cos \alpha}.$$

In the figure there is a right triangle. The hypotenuse of the triangle is the sum of the unknown radius r and the known height h of the mountain. The right angle occurs at the horizon, and its distance to the center of the earth is one side of the triangle of length r. The angle of depression α is also equal to the angle at the center of the earth, so we have a right triangle where the adjacent side is r and the hypotenuse is $r + h$. Therefore,

$$\cos \alpha = \frac{r}{r + h}.$$

Therefore,

$$r \cos \alpha + h \cos \alpha = r,$$

so

$$h \cos \alpha = r(1 - \cos \alpha).$$

Therefore,

$$r = \frac{h \cos \alpha}{1 - \cos \alpha}.$$

[Part 2.] Al-Biruni performed this measurement in a particular case, determining that $\alpha = 0°34'$ as measure from the summit of a mountain of height 652;3,18 cubits. Calculate the radius of the earth in cubits. Assuming that a cubit equals $18''$, convert your answer to miles and compare to a modern value. Comment on the efficacy of al-Biruni's procedure.

The height h is given in sexagesimal notation. 652;3,18 cubits equals 652.055 cubits decimally. That's 11737 inches, 815.07 feet, or 0.18524 miles. The angle α is also given sexagesimally. $0°34'$ equals 0.56667 degrees. Its cosine is $\cos \alpha = 0.99995109$, so $1 - \cos \alpha = 0.000048908$.

$$\begin{aligned} r &= \frac{h \cos \alpha}{1 - \cos \alpha} \\ &= \frac{0.18524 \cdot 0.99995}{0.000048908} \\ &= 3787 \end{aligned}$$

How close is 3787 miles to the actual radius? The earth isn't a perfect sphere, but its average radius is 3959 miles. Al-Biruni's estimate was only 172 miles off, that is, only about 4% short.

[Part 3.] Comment on the efficacy of al-Biruni's procedure.

It's amazingly accurate considering that it's very hard to measure a small angle like $34'$ accurately. Change that figure by a few minutes and the estimate changes a great deal.

There are more subtile things that introduce inaccuracies such as the refraction of light near the earth's surface. Also, the actual length of an ancient cubit is only known approximately. And, of course, the measurement of the height of the mountain relative to the horizon is an approximation. Incidentally, the mountain he used was at Nadana, in Pind Dadan Khan, Punjab, Pakistan. The horizon there is a plain and there's no way you could give the height of a mountain with as much accuracy as 652;3,18 cubits since the horizon varies by many tens of cubits.

Exercise 1. From Alcuin's *Propositions for Sharpening Youths*. A cask is filled to 100-*metreta* capacity through three pipes. One-third of its capacity plus 6 *modii* flows in through one pipe; one-third of its capacity flows in through another pipe; but only one-sixth of its capacity flows in through the third pipe. Haw many *sextarii* flow in through each pipe? (Here a *metreta* is 72 *sextarii* and a *modius* is 200 *sextarii*.)

Note that the cask holds 100 *metreta*, and that equals 7200 *sextarii*. The first pipe supplies $\frac{1}{3}$ of that, 2400 *sextarii*, plus 6 *modii*, which equals 1200 *sextarii*, so 3600 *sextarii* in all. The second supplies $\frac{1}{3}$ of the 7200 *sextarii*, which is 2400 *sextarii*. The third supplies $\frac{1}{6}$ of the 7200 *sextarii*, which is 1200 *sextarii*.

Exercise 17. Prove Proposition 33 of the *Maasei Hoshev*:

$$\begin{array}{rcl} (1+2+3+\cdots+n) & + & \\ (2+3+\cdots+n) & + & \\ (3+\cdots+n) & + & \\ \cdots & + & \\ (n) & = & 1^1+2^2+3^2+\cdots+n^2) \end{array}$$

There are many ways to prove this. One is to sum the terms column by column instead of row by row. That is, regroup the terms of the sum by rewriting the columns as rows

$$\begin{array}{rc} (1) & + \\ (2+2) & + \\ (3+3+3) & + \\ \cdots & + \\ (n+n+n+\cdots+n) & \end{array}$$

and note that for the k^{th} line there are k k's giving k^2, so the sum is $1^1+2^2+3^2+\cdots+n^2$.

More in the style of Levi ben Gerson's *Maasei Hoshev* would be a proof by induction as described in class.

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