Math 114 Discrete Mathematics

First Midterm Anaswers

February 2018

Scale. 90–100 A, 79–89 B, 67–78 C. Median 88.

1. Negating propositions. [15; 5 points each part] For each of the following propositions, write the negation of the proposition so that negations only appear immediately preceding predicates; there should be no negations of conjunctions, disjunctions, or quantifiers.

a. $P(x) \wedge P(y)$

The negation $\neg(P(x) \wedge P(y))$ can be converted using De Morgan's laws to $\neg P(x) \vee \neg P(y)$.

b. $\forall x\, P(x)$

Passing the negation in $\neg\forall x\, P(x)$ past the quantifier changes it from a universal quantifier to an existential one: $\exists x\, \neg P(x)$.

c. $\exists x\, \forall y\, (P(x) \to Q(x, y))$

Passing the negation past the quantifiers in $\neg\exists x\, \forall y\, (P(x) \to Q(x, y))$ yields $\forall x\, \exists y\, \neg(P(x) \to Q(x, y))$. Then, the negation of the implication $(P(x) \to Q(x, y)$ gives $P(x) \wedge \neg Q(x, y)$. So the final answer is $\forall x\, \exists y\, (P(x) \wedge \neg Q(x, y))$.

2. On truth tables. [20; 10 points each part]

a. Use a truth table to determine whether $(p \wedge q) \to r$ is logically equivalent to $(p \to r) \vee (q \to r)$. Explain in a sentence why your truth table says whether they are logically equivalent or not.

p	q	r	$(p \wedge q)$	$\to r$	$(p \to r)$	$\vee$	$(q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	T	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns for the two propositions are the same in all eight rows, therefore the propositions are logically equivalent.

b. Use a truth table to determine whether $(p \vee q \to r) \to ((p \to r) \wedge (q \to r))$ is a tautology, a contradiction, or a contingent proposition. Explain in a sentence why your truth table shows whether it is a tautology, a contradiction, or a contingent proposition.

p	q	r	$(p \vee q$	$\to r)$	$\to$	$((p \to r)$	$\wedge$	$(q \to r))$
T	T	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	F
T	T	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	T
T	T	T	T	T	T	T	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T
F	T	F	F	T	T	T	T	T

This is a tautology since it's true in all eight rows.

3. Interpretation of symbolic expressions. [25; 5 points each part] True/false.

a. $\forall x\, (x^2 > 0)$. *False.* When $x = 0$, it is not the case that $x^2 > 0$.

b. $\forall y\, \exists x\, (x < y)$. *True.* Given any number y, there is a smaller number x, for example, $x = y - 1$.

c. $\forall x\, \exists y\, (y^2 - x^2 = 1)$. *True.* Take $y = \pm\sqrt{1 + x^2}$.

d. $\exists y\, \forall x\, (y^2 - x^2 = 1)$. *False.* Given y, there are at most two values of x that satisfy this equation, $x = \pm\sqrt{y^2 - 1}$.

e. $\exists x\, \exists y\, (y^2 - x^2 = 1)$. *True.* For example, $x = 0$ and $y = 1$.

4. On proofs. [15] Prove that for any positive integer n, if 3 divides n^2, then 3 divides n. Here, "divides" means divides without remainder. [Suggestion: one way you can do this is by a proof by contradiction using cases. There are 3 cases. Case a: 3 divides n without remainder (which you're trying to show). Case b: there is a remainder of 1 when 3 divides n. Case c: there is a remainder of 2 when 3 divides n.]

The proof that follows is based on the suggestion. There are various ways it can be written up.

Proof: Suppose that 3 divides n^2. One of the three cases a, b, and c described in the suggestion holds.

Case b: Suppose there is a remainder of 1 when 3 divides n. Then $n = 3k + 1$ for some integer k. Then $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1$. Since 3 divides $9k^2 + 6k$, therefore when 3 divides n^2 there is a remainder of 1. So case b cannot occur when 3 divides n^2.

Case c: Suppose there is a remainder of 2 when 3 divides n. Then $n = 3k + 2$ for some integer k. Then $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4$. Since 3 divides $9k^2 + 12k + 3$, therefore when 3 divides n^2 there is a remainder of 1 again. So case c cannot occur when 3 divides n^2.

That leaves only case a in which 3 divides n.

Therefore, if 3 divides n^2, then 3 also divides n. Q.E.D.

There are also proofs that use other properties of positive integers such as the unique factorization theorem. We'll discuss that later in chapter 3.

5. On sets. [25] True/false.

a. If $A = \{1, 3, 5, 7, 9\}$, then $|\wp(A)| = 16$. (Recall that $\wp(A)$ is the powerset of A.) *False.* $|\wp(A)| = 2^{|A|} = 2^5 = 32$.

b. $A \cap B \subseteq C$ implies $A \cup B \cup C \subseteq A \cup B$. *False.* One counterexample is given by $A = \{1\}$, $B = \{2\}$, $C = \{1, 2, 3\}$.

c. If $A \cup B = B$, then $A \cap B = A$. *True.* Both equations are equivalent to $A \subseteq B$.

d. The composition of two onto functions is also an onto function. (Recall that a onto function is also called a surjection.) *True.*

e. If $A = B$, then $|A| = |B|$. (Recall that $|A|$ is the cardinality of the set A.) *True.* There's only one set here, and it has the same cardinality as itself.