Section 1.2, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

2. Show that $\neg(\neg p)$ and p are logically equivalent.

First, let's see a wordy explanation.

Is $\neg(\neg p) \longleftrightarrow p$ a tautology? Does it come out true no matter what truth value p has? There are two cases. In one case, p is T, so $\neg p$ is F, and $\neg(\neg p)$ is T; and since $\neg(\neg p)$ and p have the same truth value, $\neg(\neg p) \longleftrightarrow p$ comes out T. In the other case, p is F, so $\neg p$ is T, and $\neg(\neg p)$ is F; and since $\neg(\neg p)$ and p again have the same truth value, $\neg(\neg p) \longleftrightarrow p$ comes out T in this case, too. Thus, in both cases, $\neg(\neg p) \longleftrightarrow p$ comes out T. Thus, $\neg(\neg p) \longleftrightarrow p$ a tautology. Therefore, $\neg(\neg p)$ and p are logically equivalent.

That was a wordy explanation. You probably gave a much shorter one that's just as good, something like this: Negating interchanges the two truth values, so negating a second time interchanges them back to their original truth values. Since $\neg\neg p$ has the same truth value as p, therefore $\neg(\neg p) \longleftrightarrow p$ is a tautology.

Another thing you could do is present a truth table like this:

p	$\neg p$	$\neg\neg p$	$\neg(\neg p) \longleftrightarrow p$
T	F	T	T
T	T	F	T

Then, since the last column only contains Ts, it's a tautology.

6. Use a truth table to verify this De Morgan's law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

p	q	$\neg$	$(p \wedge q)$	$\longleftrightarrow$	$\neg p$	$\vee$	$\neg q$
T	T	F	T	T	F	F	F
T	F	T	F	T	F	T	T
F	T	T	F	T	T	T	F
F	F	T	F	T	T	T	T

Since $(p \wedge q) \longleftrightarrow \neg p \vee \neg q$ is T in all cases, therefore $(p \wedge q) \equiv \neg p \vee \neg q$.

You could stop one step earlier by noticing that since the columns for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are identical, therefore they're logically equivalent.

12. Show that each implication in Exercise 10 is a tautology without using truth tables.

For these, you can use the logical equivalences given in tables 6, 7, and 8.

a) $[\neg p \wedge (p \vee q)] \to q$. The following is a list of logically equivalent expressions. Since the last is a tautology, so is the first. Each step uses one of the logical equivalences in one of the tables to substitute one subexpression for a logically equivalent subexpression.

$$\begin{array}{rcl}
[\neg p \wedge (p \vee q)] & \to & q \\
\neg[\neg p \wedge (p \vee q)] & \vee & q \\
(\neg\neg p \vee \neg(p \vee q)) & \vee & q \\
(p \vee (\neg p \wedge \neg q)) & \vee & q \\
((p \vee \neg p) \wedge (p \vee \neg q)) & \vee & q \\
(T \wedge (p \vee \neg q)) & \vee & q \\
(p \vee \neg q) & \vee & q \\
p & \vee & (\neg q \vee q) \\
p & \vee & T \\
 & T &
\end{array}$$

There are many other routes you could take to reduce the original expression to T. This was just one of them.

The other parts of 10 are similar. Here's how 10c might be proved.

$$\begin{array}{rcl}
p \wedge (p \to q) & \to & q \\
p \wedge (\neg p \vee q) & \to & q \\
(p \wedge \neg p) \vee (p \wedge q) & \to & q \\
F \vee (p \wedge q) & \to & q \\
p \wedge q & \to & q \\
\neg(p \wedge q) & \vee & q \\
(\neg p \vee \neg q) & \vee & q \\
\neg p & \vee & (\neg q \vee q) \\
\neg p & \vee & T \\
 & T &
\end{array}$$

14. Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology. The easiest way is simply to use a truth table.

p	q	$(\neg p$	$\wedge$	$(p \to q))$	$\to$	$\neg q$
T	T	F	F	T	T	F
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

You'll note that the third row does not have a T in the $\to$ column, so it's not a tautology.

Instead of using a truth table, you could consider the single case when p is F and q is T, and show that $(\neg p \wedge (p \to q)) \to \neg q$ comes out F.

20. Show that $\neg(p \oplus q)$ and $p \longleftrightarrow q$ are logically equivalent. This is an important logical equivalence and well worth memorizing. The proof is easy by a truth table and is omitted here.

35. Find the dual of each of the following propsitions.

a) $p \wedge \neg q \wedge \neg r$. The dual is $p \vee \neg q \vee \neg r$. Just turn all the $\wedge$'s into $\vee$'s.

b) $(p \wedge q \wedge r) \vee s$. The dual is $(p \vee q \vee r) \wedge s$. Just interchange $\wedge$'s and $\vee$'s.

c) $(p \vee F) \wedge (q \vee T)$. The dual is $(p \wedge T) \vee (q \wedge F)$. Besides interchanging $\wedge$'s and $\vee$'s, be sure to interchange T's and F's, too.

46 and 48. Construct truth tables for NAND and NOR.

p	q	p NAND q	p NOR q
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

50. Show that NOR, denoted $\downarrow$, is functionally complete. As described above problem 43, a collection of logical operators is called *functionally complete* if every compound propsition is logically equivalent to a compound proposition involving only logical operators in the collection. In problem 43, the three logical operators $\wedge, \vee$, and $\neg$ were shown to be functionally complete. All we have to do is show that these three operators can each be described in terms of $\downarrow$. Indeed, by problem 43, we only have to consider two of these operators.

a) Show that $p \downarrow p$ is logically equivalent to $\neg p$. Just use a truth table.

b) Show that $(p \downarrow q) \downarrow (p \downarrow q)$ is logically equivalent to $p \wedge q$. Again, a truth table is the simplest way.

c) Since problem 44 shows that $\neg$ and $\wedge$ form a functionally complete collection of logical operators, and each of these can be written in terms of $\downarrow$, therefore $\downarrow$ by itself is a functionally complete collection of logical operators.

One implication of this result is that all the logical ciruitry of a computer can be constructed from only one kind of logical gate, a nor-gate.

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