Section 1.7, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

12. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

Let $a = m/n$ and $b = j/k$ where m, n, j, and k are integers. Is the expression

$$(m/n)^{j/k} = \sqrt[k]{(m/n)^j}$$

always rational? It is if $k = \pm 1$, not always otherwise. Take, for example, $k = 2$, $j = 1$, $a = 2$. Then $a^b = \sqrt{2}$, which we know is not rational.

Thus, we've disproved the statement by finding a counterexample.

22. The *quadratic mean* of two real numbers x and y equals $\sqrt{(x^2 + y^2)/2}$. Formulate a conjecture about the relative sizes of arithmetic and quadratic means of positive numbers, and prove your conjecture.

A small table of values probably enough. It would be a good idea to use some numbers that are very large and others that are very small. You'll probably note very soon that if $x = y$ then the two means are the same. Also, if x is near 0 while y is very large, then the arithmetic mean is about $y/2$ while the quadratic mean is about $y/\sqrt{2}$, which is larger. After a while you probably conjectured that the quadratic mean is no less than the aritmetic mean,

$$\sqrt{(x^2 + y^2)/2} \geq (x + y)/2.$$

Next, to prove it, you likely reduced the problem to a simpler one. Since all the numbers involved are positive, the required inequality can be squared to get an equivalent inequality.

$$(x^2 + y^2)/2 \geq (x + y)^2/4$$

More algebraic operations can be applied to the inequality to simplify it further, first $2x^2 + 2y^2 \geq x^2 + 2xy + y^2$, then

$$x^2 + y^2 \geq 2xy.$$

At some point in your investigation, you may have needed a little inspiration to continue. You might have written the last inequality as

$$x^2 - 2xy + y^2 \geq 0$$

and noticed that the term $x^2 - 2xy + y^2$ equals $(x - y)^2$. But

$$(x - y)^2 \geq 0$$

since the square of every real number is nonnegative. Thus, by a series logical equivalences, you reduced the inequality to a known inequality, thus proving the theorem.

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