Section 3.5, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

4. Find the prime factorization of each of the following.

a. $39 = 3 \cdot 13$.

b. $81 = 3^4$.

c. 101 is prime! You only have to check that 2, 3, 5, and 7 don't divide it.

d. $143 = 11 \cdot 13$.

e. $289 = 17^2$.

f. 899 is prime. Well, let's check the first few primes to see if any of them divide it. A calculator might help here to save time. 2, 3, 5, 7, 11, 13, 17, 19, and 23 all don't divide 899. But 29 does. $899 = 29 \cdot 31$.

14. A number is *perfect* if it equals the sum of its proper divisors.

a. Show that 6 and 28 are perfect. $6 = 1+2+3$. $28 = 1 + 2 + 4 + 7 + 14$.

b. Show that $2^{p-1}(2^p - 1)$ is perfect when $2^p - 1$ is prime. Suppose $2^p - 1$ is prime. Then its only factors are 1 and itself. The factors of 2^{p-1} are the powers of 2 from 2^0 through 2^{p-1}. Since 2^{p-1} and $2^p - 1$ are relatively prime, therefore the factors of $2^{p-1}(2^p-1)$ are (1) the powers of 2 from 2^0 through 2^{p-1}, and (2) those numbers times $2^p - 1$. So the proper factors are

$$1, 2, 4, \ldots, 2^{p-1}$$

and

$$2^p - 1, 2(2^p - 1), 4(2^p - 1), \ldots, 2^{p-2}(2^p - 1).$$

The first row is a geometric series whose sum is $2^p - 1$. The second is also a geometric series whose sum is $(2^{p-1} - 1)(2^p - 1)$. Adding these two sums together gives $2^{p-1}(2^p - 1)$. Thus, $2^{p-1}(2^p - 1)$ is the sum of its proper divisors, and so it is a perfect number.

20. What are the greatest common divisors of the following pairs of integers?

a. $2^2 \cdot 3^3 \cdot 5^5$ and $2^5 \cdot 3^3 \cdot 5^2$. For the powers of each prime take the minimum of the power in the first and in the second number. For example, in the first number 2 appears to the power 2, but in the second it appears to the power 5, so in the GCD 2 will appear to the power 2, the minimum of 2 and 5. The GCD is $2^2 \cdot 3^3 \cdot 5^2$.

b. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$. If a prime only appears in one of the numbers, then it won't appear in the GCD. Answer: $2 \cdot 3 \cdot 11$.

c. 17 and 17^{17}. The first divides the second, so it will be the GCD.

d. $2^2 \cdot 7$ and $5^3 \cdot 13$. These numbers are relatively prime, so the GCD is 1.

e. 0 and 5. Since 5 divides 0, it's the GCD.

f. $2 \cdot 3 \cdot 5 \cdot 7$ and itself. The GCD of any number and itself is just itself.

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