Math 114 Discrete Mathematics

Section 4.1, selected answers

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9. Find a formula for the sum of the first n even positive integers, and use mathematical induction to prove your formula is correct.

n	1	2	3	4	5	6
$2n$	2	4	6	8	10	12
Σ	2	6	12	20	30	42

It appears that the sum $\displaystyle\sum_{k=1}^{n} 2k$ is $n^2 + n$.

To prove it using math induction, first check the base case. Is the sum of the first $n = 1$ even number (which is just 2) equal to $1^2 + 1$? Yes, it is.

Next, the inductive step. Assume the inductive hypothesis $P(n)$:

$$\sum_{k=1}^{n} 2k = n^2 + n$$

is true, and prove that $P(n+1)$:

$$\sum_{k=1}^{n+1} 2k = (n+1)^2 + (n+1)$$

is also true. Starting with the LHS

$$\sum_{k=1}^{n+1} 2k = \left(\sum_{k=1}^{n} 2k\right) + 2(n+1)$$

which, by the inductive hypothesis,

$$= (n^2 + n) + 2(n+1)$$

With a little algebra, that can be written as the RHS:

$$= n^2 + 2n + 1 + n + 1 = (n+1)^2 + (n+1).$$

Therefore, the inductive step is valid and the proof is complete.

10. Find a formula for

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n. Use mathematical induction to prove your result.

Start by doing the additions, not with a calculator, but by hand. If you use a calculator, you probably won't see the pattern.

$$\begin{aligned} \frac{1}{1\cdot 2} &= \frac{1}{2} \\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} &= \frac{2}{3} \\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} &= \frac{3}{4} \end{aligned}$$

Aha! The pattern suggests that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Now, to prove it. The first line shows it's true for the base case.

Next, the inductive step. Assume the inductive hypothesis

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

and prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

By the inductive hypothesis, the LHS

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}.$$

But that simplifies to

$$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)},$$

and that reduces to the RHS.

Therefore, the inductive step is valid, and the proof is complete.

14. Prove that for every positive integer n,

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.$$

The base is when $n = 1$. In that case, the equation says that $1 \cdot 2^1 = (1-1)2^{1+1} + 2$, which is true.

For the inductive step, we assume the inductive hypothesis that for a given value of n,

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2,$$

and we have to show that for the next value of n, namely $n+1$, that

$$\sum_{k=1}^{n+1} k2^k = ((n+1)-1)2^{(n+1+1} + 2,$$

that is, that

$$\sum_{k=1}^{n+1} k2^k = n2^{(n+2} + 2.$$

16. Use mathematical induction to prove that
$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2)$
$= n(n+1)(n+2)(n+3)/4$.

Let's denote the left hand side of this equation by LHS(n) and the right hand side by RHS(n).

You have to decide what the base case is. Is it $n = 0$ or $n = 1$? It isn't clear from the statement of the problem. If you take $n = 0$ to be the base case, then LHS(0) is an empty sum, and an empty sum is 0. But RHS(0) is $0 \cdot 1 \cdot 2 \cdot 3/4$, which is also 0. The other choice for a base case is $n = 1$. Then LHS(1) is $1 \cdot 2 \cdot 3$, while RHS(1) $1 \cdot 2 \cdot 3 \cdot 4/4$, so the two sides are equal. So, the base case is taken care of either with $n = 0$ or with $n = 1$.

Now for the inductive step. Suppose that the statement is true for n, that is, LHS(n) = RHS(n). We have to prove that it's true for $n+1$, that is, LHS($n+1$) = RHS($n+1$), which written is full is $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (n+1)(n+2)(n+2) = (n+1)(n+2)(n+3)(n+4)/4$.

The only difference between the LHS(n)and LHS($n+1$) is that the latter has one more term, namely, $(n+1)(n+2)(n+2)$. That is,

$$\text{LHS}(n+1) - \text{LHS}(n) = (n+1)(n+2)(n+2).$$

If we can show that RHS($n+1$) − RHS(n) has the same value, then we may conclude LHS(n) = RHS(n) implies LHS($n+1$) = RHS($n+1$). But

$$\begin{aligned}
& \text{RHS}(n+1) - \text{RHS}(n) \\
= \; & (n+1)(n+2)(n+3)(n+4)/4 - \\
& \quad n(n+1)(n+2)(n+3)/4 \\
= \; & \big((n+4) - n\big)(n+1)(n+2)(n+3)/4 \\
= \; & (n+1)(n+2)(n+3)
\end{aligned}$$

That finishes the inductive step, so we've finished the proof.

21. Show that $2^n > n^2$ whenever n is an integer greater than 4.

In this case, the base case occurs when $n = 5$, so you need to check that $2^5 > 5^2$, which, of course, is true.

Now for the inductive step. Assume the inductive hypothesis $2^n > n^2$, where $n > 4$, and prove that $2^{n+1} > (n+1)^2$.

$$\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&> 2n^2
\end{aligned}$$

At this point, it suffices to show that $2n^2 > (n+1)^2$, and that's logically equivalent to the inequality $n^2 > 2n+1$, or $n^2 - 2x + 1 > 2$, that is, $(n-1)^2 > 2$. But $n > 4$, so $(n-1)^2 > 9 > 2$. Thus, the inductive conclusion $2^{n+1} > (n+1)^2$ follows, and the proof is complete.

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