Notes on combinatorics

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What is combinatorics? It's that part of discrete mathematics devoted to counting things. The things are often sets with some kind of structure, perhaps subsets of other sets with certain properties. There are various principles used in counting and we'll look at some of the more basic ones.

The multiplicative principle, choices and stages, and tree diagrams. Suppose you're making choices in several stages and the number of choices m_i you can choose from at a stage i doesn't depend on previous choices you've made. Then the total number of outcomes for n stages $1, 2, \ldots, n$ is the product $m_1 m_2 \ldots m_n$.

We'll clarify this multiplicative principle with a couple of examples and illustrate them with tree diagrams. Some of the primary applications of this multiplicative principle are to counting permutations and combinations.

A special case of this is the product of sets. Given finite sets $A_1, A_2, \ldots, A_n$, their product $A_1 \times A_2 \times \cdots \times A_n$ consists of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where each a_i belongs to the corresponding set A_i. In order to choose one of these ordered n-tuples, for the first stage you have a choice of choosing any one of the elements of A_1 to be a_1. The number of choices at stage 1 is the number of elements of $A_1 4$, usually called the *cardinality* of A_1 and denoted $|A_1|$. For the second stage you have $|A_2|$ choices for a_2, and so forth. The multiplicative principle gives us the standard formula for the cardinality of the product

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|$$

Permutations. There are a couple of different ways you can describe permutations. One fairly abstract definition is that a *permutation* of a set A is a one-to-one mapping σ of A to itself. That means that σ is a function $A \to A$ that has an inverse function σ^{-1} such that

$$\sigma(x) = y \quad \text{if and only if} \quad x = \sigma^{-1}(y).$$

For example, take $A = \{a, b, c\}$ and let σ be the permutation $\sigma(a) = b$, $\sigma(b) = c$, and $\sigma(c) = a$. It would be nice to have a more concise notation for permutations than describing what it does to each element of the set A one at a time. There are a couple of useful notations. One is to write it as a two-line table:

$$\sigma = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}.$$

Another is to specify a standard ordering for the elements of A–such as *abc*–and just list the second line of the the two-line table: *bca*. That describes the permutation as a rearrangement.

Counting permutations. Our main question, an easily answered one, is how may permutations are there on a set A of n elements? If n is small, say 4, then we can list all the permutations. Let's list all the rearrangements of *abcd*.

abcd	*bacd*	*cabd*	*dabc*
abdc	*badc*	*cadb*	*dacb*
acbd	*bcad*	*cbad*	*dbac*
acdb	*bcda*	*cbda*	*dbca*
adbc	*bdac*	*cdab*	*dcab*
adcb	*bdca*	*cdba*	*dcba*

There are 24 of them.

Even when n is not small, it's easy to determine how many permutations there are. We just use the multiplicative principle. In the first stage, choose one of the n elements to go first. In the second stage, there are $n-1$ remaining elements, and choose one of them to go second. At the next stage, chose one of the remaining $n-2$ elements to go next. And so forth until the last stage, when there's only one element left, so it goes last. Thus, the number of permutations of a set of n elements is

$$n(n-1)(n-2)\cdots 2\cdot 1.$$

This last expression is usually abbreviated $n!$ and read "n factorial" or "factorial n" (except by some people who like to say "n shriek" or "n bang").

Thus, there are $4! = 24$ permutations of a set of 4 elements; $3! = 6$ permutations of a set of 3 elements; $2! = 2$ permutations of a set of 2 elements; $1! = 1$ permutations of a set of 1 element; and $0! = 1$ permutations of the empty set $\emptyset$. The last is because the unique function $\emptyset \to \emptyset$ is, by our definition, a permutation.

Sterling's approximation for factorials. The factorial function $n!$ grows very fast with n. It's approximately $n^n e^{-n}\sqrt{2\pi n}$. That's Sterling's approximation. Sterling's approximation is particularly useful when you're examining behavior for large n.

k-permutations. Sometimes we don't want full permutations of a set of n elements, but just partial permutations. If $k \le n$, a *k-permutation* is an ordered listing of just k elements of a set of n elements. For instance, the 3-permutations of $abcd$ are these

abc	bac	cab	dab
abd	bad	cad	dac
acb	bca	cba	dba
acd	bcd	cbd	dbc
adb	bda	cda	dca
adc	bdc	cdb	dcb

while the 2-permutations are these

ab	ba	ca	da
ac	bc	cb	db
ad	bd	cd	dc

We can determine how many k-permutations of a set of n elements there are using the multiplicative principle. In the first stage, choose one of the n elements to go first. In the second stage, there are $n-1$ remaining elements, and choose one of them to go second. At the next stage, chose one of the remaining $n-2$ elements to go next. And so forth until the kth stage, when there are $n-k+1$ remaining elements. Thus, the number of k-permutations of a set of n elements is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

There is no particular standard notation for the number of k-permutations of a set of n elements, but you'll see it denoted $(n)_k$ as in our text, nPk, P^n_k, and various other things. We'll probably just use $n!/(n-k)!$.

Today. Combinations, binomial coefficients, and Pascal's triangle.

Definition of combinations and their relation to partial permutations. A *combination* of size k from a set S of size n is just a subset of size k, and it's more often it's called a k-subset.

A k-subset is related to k-permutations but they're not the same. A k-permutation is a listing of k distinct elements of S where the order of the elements in the listing is relevant. But for a k-subset, the elements are not listed in any particular order; that is, order doesn't matter.

Let's take an example. Let S be the 5-element set $S = \{a,b,c,d,e\}$. There are $5\cdot 4\cdot 3 = 60$ 3-permutations of S, but there are far fewer 3-subsets of S. For instance, one 3-subset is $\{a,b,c\}$. But this subset is associated to 6 of the 3-permutations, namely, abc, acb, bac, bca, cab, and cba. There are 6, of course, because there are $3! = 6$ full permutations of a set of 3 elements.

In general, each k-subset is associated to $k!$ of the k-permutations. Since there are $\dfrac{n!}{(n-k)!}$ of the k-permutations altogether, that implies that the number of k-subsets of a set of n elements is exactly $\dfrac{n!}{k!(n-k)!}$.

Binomial coefficients. That last expression is called a *binomial coefficient*, and it's denoted $\dbinom{n}{k}$, pronounced "n choose k".

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Thus, there are $\binom{n}{k}$ subsets of size k in a set of size n.

Note that this definition is relevant so long as $0 \leq k \leq n$ and $n \geq 0$, where, as always, $0! = 1$.

Pascal's triangle. We'll compute a few of these binomial coefficients. Then we'll show the recurrence relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

then display these binomial coefficients in a triangular table usually called *Pascal's triangle*. This triangle long predates Pascal, but it's named for him because he developed many of its properties and used it to answer questions in probability. Here's the top few rows of the triangle.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The numbers along the sides are all 1s, and each entry in the middle is the sum of the two entries above it. These are just the binomial coefficients $\binom{n}{k}$ arranged in a table. By making it a triangle rather than a rectangle, you can see the two relationships $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and $\binom{n}{k} = \binom{n}{n-k}$ more clearly. Here's the triangle again, but one less row of it so it fits on the page.

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

There are lots of interrelations among these entries in Pascal's triangle, and we may have time to look at a couple of them. Note that in each row, n is fixed. Let's call that the nth row; the top row is then the 0th row. Note that the numbers in the nth row sum to 2^n.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

For instance, when $n = 5$, we have $1+5+10+10+5+1 = 32 = 2^5$. That's because these binomial coefficients tell us the number of subsets of various sizes of a set of n elements. Since there are 2^n subsets in all, they have to add up to 2^n.