Math 114 Discrete Math Final

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You may use a sheet of notes and a calculator during this test. Write your answers on the test sheets. Show all your work for credit.

Points for each problem will be in square brackets. For all problems, numerical expressions are fine for answers, for instance, you may answer $2^8 - 1$ instead of 255 if you like.

Write your answers in a bluebook. You can do the problems in any order you like, but start each problem on a separate page.

Problem 1. On the basics of counting [16] Consider strings of letters over the standard 26-letter alphabet.

a. How many strings of length 4 do not have the letter `x` in them?
b. How many strings of length 4 have the letter `x` exactly twice?
c. How many strings of length 4 do not include any letter twice?
d. How many strings of length 4 contain the substring `xy` at least once? (That is, how many have the letter `x` immediately before the letter `y`?)

Problem 2. On the pigeonhole principle. [10] There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

Problem 3. On binomial coefficients. [10] Recall the binomial theorem which states that

$$(x+y)^n = \sum_{k \to 0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} + \cdots \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Prove that the entries in the n-th row of Pascal's triangle sum to 2^n, that is, $\displaystyle\sum_{k \to 0}^{n} \binom{n}{k} = 2^n$.

Problem 4. On mathematical induction. [12] Use mathematical induction to prove that

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3},$$

that is, that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3.$$

In your proof, identify the basis step and the inductive step.

Problem 5. On relations. [10] Give an example of a relation on a set that is

a. symmetric and reflexive, but not transitive.

b. reflexive and transitive, but not symmetric.

Problem 6. On methods of proof. [15] Recall that a rational number is any number of the form m/n where m and n are both integers, but n is not 0.

a. Prove or disprove: the sum of two rational numbers is rational.

b. Prove or disprove: the sum of two irrational numbers is irrational

c. For each of a and b above, what method of proof did you use? Here are some possible answers: direct proof, indirect proof, proof by induction, proof by case, example/counterexample.

Problem 7. On discrete probability. [12] Suppose that a random permutation is selected from the set $\{1, 2, 3, 4, 5\}$. Explain your answers.

a. What is the probability that 2 precedes 3?

b. What is the probability that 2 immediately precedes 3?

c. What is the probability that 2 precedes 3 and 4 precedes 5?

Problem 8. On equivalence relations. [15] Let R be the relation on ordered pairs of positive integers such that $(a, b)R(c, d)$ if and only if $ad = bc$. So, for example $(3, 4)R(6, 8)$, but not $(3, 4)R(6, 9)$. Note that R is a reflexive relation: $(a, b)R(a, b)$ since $ab = ba$ for all positive integers a and b. In fact, R is an equivalence relation as you will show in parts a and b.

a. Prove that R is a symmetric relation.

b. Prove that R is a transitive relation.

c. Describe the equivalence class of $(4, 6)$.