The growth of functions
Math 214 Discrete Mathematics

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I've found that approaching the topic of the growth of functions via Landau's Big-$\mathcal{O}$ notation is very confusing. Hardy's $\prec$ notation makes it much easier to understand. We'll start with the same concept of when one function "dominates" another and treat that as a partial ordering of functions. It's entirely the same thing, but the notation is more appropriate.

Definition. Let f and g be two functions from the positive real numbers to the positive real numbers. We say that f is *dominated* by g, written $f \preceq g$, if some multiple of g is eventually greater than f. More precisely, there is a constant C (the multiple) and a constant k (which indicates the "eventually") so that for all $x > k$

$$f(x) \leq Cg(x).$$

In the big-$\mathcal{O}$ terminology, we say that f is $\mathcal{O}(g)$, or g is $\Omega(f)$. (Knuth popularized the Ω notation and the Θ notation mentioned below.)

Notes on the definition. Since the condition only applies to large values of x, it isn't necessary for f and g to be defined on all the positive real numbers. It's enough that they're eventually defined. In fact, it's enough that they both are defined for the same infinite set of numbers, say positive integers. Furthermore, there's a way to extend the definition so that the values of f and g can be negative, but that's not important since we're only interested in positive valued functions.

We'll look at some examples, including their graphs, in class.

The constant C allows us to ignore constant multiples and treat, for instance, the function $5x^2$ the same as the function x^2.

The constant k allows us to ignore small values of x. For example, although x is not always smaller than x^2, it is if x is greater than $k = 1$.

Definition. If f and g are each dominated by the other, that is, both $f \preceq g$ and $g \preceq f$. we say that they have the same *order* of growth, written $f \simeq g$. In the big-$\mathcal{O}$ terminology, we say that f is $\Theta(g)$. If g dominates f, but f doesn't dominate g, we'll say f is *strictly dominated* by g, written $f \prec g$, or in Landau's terminology f is $o(g)$, f is little-oh of g.

Theorem. Dominance is a reflexive and transitive relation, that is,

$\forall f, f \preceq f$

$\forall f\, \forall g\, \forall h, f \preceq g \wedge g \preceq h \Rightarrow f \preceq h$

Furthermore, having the same order is an equivalence relation, which means it is reflexive, symmetric, and transitive.

$\forall f, f \simeq f$

$\forall f\, \forall g\, \forall h, f \simeq g \wedge g \simeq h \Rightarrow f \simeq h.$

Strict dominance, $\prec$, is also a reflexive and transitive relation.

The proofs are straightforward, and we'll look at a couple of them in class. The concepts allow us to group functions that have similar growth together, but separate those that have very different growths. Some functions that have different growth rates are

$$1 \prec \log n \prec \sqrt{n} \prec n \prec n \log n \prec n^2 \prec n^3 \prec 2^n \prec n!$$

There are a number of theorems that help us classify orders of growth. Some of them show that the three relations, $\preceq$, $\simeq$, and $\prec$, act like the three relations, $\leq$, $=$, and $<$, but some are specific to the dominance relations.

Theorem. If $h \prec g$, then $g \pm h \simeq g$.

This theorem allows us to ignore low order terms whenever a function is a sum of terms. For example, if $f(x) = x^3 + 3x^2 + 5$, then $f(x) \simeq x^3$ since the other two terms have lower order. In particular, we have the following corollary.

Theorem. A polynomial of degree n has the same order as x^n.

Theorem. If $f \preceq g$ and $h \preceq k$, then $fh \preceq gk$. Likewise, if $f \simeq g$ and $h \simeq k$, then $fh = gk$. Also, if $f \prec g$ and $h \preceq k$, then $fh \prec gk$.

With these theorems we can quickly find the order of a function. For instance, consider the function $x \log(x^2 + 1) + x^2 \log x$. Since $\log(x^2 + 1) \simeq \log x$, therefore $x \log(x^2 + 1) + x^2 \log x \simeq x \log x + x^2 \log x$. But $x \prec x^2$, so $x \log x \prec x^2 \log x$, therefore $x \log x + x^2 \log x \simeq x^2 \log x$, and we've found that the order of the original function is $x^2 \log x$.

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