Math 114, Discrete Mathematics,
Midterm, Answers. Oct, 2003.

Problem 1. Translation into symbolic expressions. [12] Let p, q, and r be abbreviations for the following propositions

p: "Grizzly bears have been seen in the area,"
q: "Hiking is safe on the trail," and
r: "Berries are ripe along the trail."

Write these propositions using p, q, and r and logical connectives.

a. Berries are ripe along the trail, but grizzly bears have not been seen in the area.

$r \wedge \neg p$. Note that "but" carries the same logical significance as "and."

b. If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been see in the area.

$r \to (q \leftrightarrow \neg p)$. Note that the parentheses are required.

c. Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

$p \wedge r \to \neg q$.

Problem 2. On truth tables. [10] Use a truth table to show that $(p \to r) \wedge (q \to r)$ is logically equivalent to $(p \vee q) \to r$.

p	q	r	$(p \to r)$	$\wedge$	$(q \to r)$	$(p \vee q)$	$\to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Since the columns are the same under the main connectives of each of the two formulas, therefore they are logically equivalent.

Problem 3. Interpretation of symbolic expressions. [15] Determine the truth value of each of the following statements if the universe of discourse for each variable consists of all real numbers. Simply write "true" or "false" for each; no need to explain why.

a. $\forall x\, \exists y\, (x = y^2)$. Does every number x have a square root y? No, negative numbers don't. So this is false.

b. $\exists x\, \exists y\, (x + y \neq y + x)$. Is it ever the case that $x + y$ doesn't equal $y + x$? No. So this is false.

c. $\forall x{\neq}0\, \exists y\, (xy = 1)$. Given a nonzero number x can you find a number y so that $xy = 1$. Yes, it's reciprocal,$y = 1/x$. So this is true.

d. $\exists y\, \forall x{\neq}0\, (xy = 1)$. Is there some number y such that if you multiply it by every nonzero number x you always get 1? No. So this is false.

e. $\forall x\, \exists y\, (2x + 3y = 7)$. Given any x can you solve this equation for y? Yes, $y = (2x - 7)/3$. So this is true.

Problem 4. On rules of inference. [10] Draw relevant conclusions from statements 1, 2, and 3 below, and state what rules of inference you use. (If you don't know the name for a rule of inference, then just state the whole rule symbolically.)

1. "If I play hockey, then I am sore the next day." $H \to S$.
2. "I use the whirlpool if I am sore." $S \to W$.
3. "I did not use the whirlpool." $\neg W$.

There are a couple ways to continue. You could combine 1 and 2 by hypothetical syllogism to get
4. $H \to W$: "If I play hockey, then I use the whirlpool."
Then continue by combining 3 and 4 by modus tollens to get
5. $\neg H$: "I didn't play hockey."

Alternatively, you could combine 2 and 3 by modus tollens to get
4′. $\neg S$: "I was not sore."
Then continue by combining 4′ and 1 to get 5.

Problem 5. On the growth of functions. [10] Which of these following functions are $O(x^3)$? Simply write "yes" or "no" for each; no need to explain why.

a. $f(x) = 4x^3 + 5x - 8$. Yes, this is $O(x^3)$, since all cubic polynomials are.

b. $f(x) = x^2 \log x$. Yes, it is. Since $\log x$ is $O(x)$, therefore $x^2 \log x$ is $O(x^2 x)$.

c. $f(x) = 2^x$. No, it's not. Exponential functions grow faster than any polynomial function.

Problem 6. On algorithm analysis. [10] Here's an algorithm that will sort an array of integers $a_1, a_2, \ldots, a_n$ of length n into increasing order. First it will be described verbally, then in pseudocode.

Scan down the entire array comparing adjacent pairs of elements, and switch them if they're out of order. Perform this whole array scan $n - 1$ times.

In the following pseudocode, comments are placed in braces {}.

For $i = 1$ to $n - 1$ {i is used to count the number of scans}

`For` $j = 1$ `to` $n-1$ $\{j$ `is used travel down the array`$\}$

`If` $a_j > a_{j+1}$ `then exchange` a_j `with` a_{j+1}

a. In this algorithm, exactly how many times will adjacent pairs of elements be compared? That is, in the pseudocode, how many times will the comparison $a_j > a_{j+1}$ be executed? (Your answer will be expressed in terms of n. Hint: how many scans? how many comparisons per scan?)

There are $n-1$ scans, and in each scan there are $n-1$ comparisons. Therefore, there are $(n-1)^2$ comparisons in all.

b. What is the time complexity of this sorting algorithm measured in terms of comparisons? That is, what is a big-O estimate for the amount of time that it will take to execute?

The function $(n-1)^2$ is $O(n^2)$.

Problem 7. On set theory and its notation. [12] Let A, B, and C be the following sets.

$$A = \{1,3,5,7,9\} \quad B = \{2,3,6,7,10\} \quad C = \{1,2,6,9\}.$$

a. The symmetric difference $A \oplus B$ of A and B is $\{1,2,5,6,9,10\}$. True.

b. $A \cap B$ is disjoint from C. True; no element that is in both of A and B also lies in C.

c. The cardinality of $A \times B$ is 10. False, it's 25.

d. The cardinality of the power set of C is 16. True, it's 2^4.

Problem 8. On modular arithmetic. [10] Find an integral solution to the congruence

$$5n \equiv 1 \pmod{11}.$$

Find a multiple of 5 that when you divide it by 11 gives a remainder of 1. Since $5 \cdot 9 = 45 = 4 \cdot 11 + 1$, therefore an answer is 9.

Problem 9. On the Euclidean algorithm. [10]

a. Use the Euclidean algorithm to determine the greatest common divisor of the two integers 667 and 437. Show your work.

First, $667 - 437 = 230$. Next, $437 - 230 = 207$. Then, $230 - 207 = 23$. Since 23 evenly divides 207, the GCD is 23.

b. Use the results of part a to reduce the fraction 437/667 to lowest terms.

Divide the GCD 23 into the numerator and denominator to conclude

$$\frac{437}{667} = \frac{19}{29}.$$