Sets and Set Operations

D. Joyce, Clark University

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Subsets and predicates. For these notes we'll look at one set U, called the *universal set*, and its subsets. In later notes, we'll build new sets out of old ones using the product construction and the powerset construction.

The universal set U corresponds to the domain of discourse in predicate logic when we're only considering unary predicates on that domain. Let, for instance, U be the set of integers, usually denoted $\mathbf{Z}$. A couple of unary predicates for this domain are $S(x)$: "x is a perfect square," and $P(x)$: "x is a positive integer." These two predicates correspond to two subsets of U. The first corresponds to the set of perfect squares which includes $0, 1, 4, 9$, etc., and the second corresponds to the set of positive integers which includes $1, 2, 3$, etc. The subset that corresponds to a unary predicate is called the *extent* of the predicate.

There's such a close correspondence between a unary predicate and it's extent that we might as well use the same symbol for both. So, we can use S for the subset of perfect squares, or S for the predicate which indicates with the notation $S(x)$ whether an integer x is a perfect square or not.

There are a couple of ways to use notation to specify a set. One is by listing its elements, at least the first few, and hoping the reader can understand your intent.

$$S = \{0, 1, 4, 9, \ldots\}.$$

That's pretty informal. Another is to use the so-called set-builder notation which explicitly makes the connection between the subset and the predicate

$$\begin{aligned} S &= \{x \in U \mid S(x)\} \\ &= \{x \in U \mid x \text{ is a perfect square}\} \end{aligned}$$

The symbols $x \in U$ indicates that x is an *element* of the set U, also called a *member* of U.

Operations on subsets. Some of the logical operations correspond directly to operations on subsets. Intersection on subsets corresponds to conjunction, union to disjunction (inclusive or), and symmetric difference corresponds to exclusive or. Besides the symbolic connections

$$\begin{aligned} A \cap B &= \{x \in U \mid x \in A \wedge x \in B\} \\ A \cup B &= \{x \in U \mid x \in A \vee x \in B\} \\ A \oplus B &= \{x \in U \mid x \in A \oplus x \in B\} \end{aligned}$$

we'll look at Venn diagrams. Venn diagrams are easier for most humans to use then algebraic symbols.

The universal complement $\overline{A}$ of a subset A corresponds to negation, but the universal set must be understood. Also, be aware the variant notations are used for universal complement, like A^c or A'. There's also a relative complement, also called difference.

$$\begin{aligned} \overline{A} &= \{x \in U \mid x \notin A\} \\ A - B &= \{x \in U \mid x \in A \vee x \notin B\} \end{aligned}$$

One special subset of U is the *empty set*, denoted $\emptyset$, also called the *null set* or *void set*. It's the complement of U

$$\emptyset = \overline{U} = \{x \in U \mid x \neq x\}.$$

Note that $\emptyset$ has no elements. Set theory could be done without the use of $\emptyset$, as Dedekind did, but it's as useful to have it in set theory as zero is useful in arithmetic.

Incidentally, the number of elements in a set is called its *cardinality*, and it's denoted with absolute value signs. Thus, $|\emptyset| = 0$. Some sets are finite and some infinite. When we discuss infinite sets later, we'll see that not all infinite sets have the same cardinality. That will require us to be more careful about the definition of cardinality.

Subsets, proper subsets, and equality of subsets. We say one subset A is a *subset* of another subset B if every element of A is an element of B. We'll use the notation $A \subseteq B$.

$$A \subseteq B \text{ iff } \forall x \in U (x \in A \to x \in B)$$

. Note that the empty set is a subset of every A, that A is a subset of itself, and that A is a subset of the universal set U.

$$\emptyset \subseteq A \subseteq A \subseteq U.$$

Although A is a subset of itself, it's special in that it's also equal to itself. We'll say that a *proper subset* B of A is a subset of A that's not equal to A, and we'll denote that by $B \subset A$. (Warning: some people use $B \subsetneqq A$ for proper subset and $B \subset A$ for subset.)

Note that the only way that A and B can each be subsets of eachother is when they are equal

$$\begin{array}{ll} & A \subseteq B \text{ and } B \subseteq A \\ \text{iff} & \forall x (x \in A \to x \in B) \land \forall x (x \in B \to x \in A) \\ \text{iff} & \forall x ((x \in A \to x \in B) \land (x \in B \to x \in A)) \\ \text{iff} & \forall x \in U, (x \in A \leftrightarrow x \in B) \\ \text{iff} & A = B \end{array}$$

Indexed unions and intersections. Sometimes it's useful to index our unions and intersections, especially where there are many operands. When we want the union of n subsets, $A_1, A_2, \ldots,$ and A_n, we could simply use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n,$$

but an alternative notation for the same union is

$$\bigcup_{i=1}^{n} A_i$$

where the indexing variable i is meant to take all the values from 1 through n. Likewise for the intersection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n.$$

We might also want an infinite union, such as

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots.$$

An element is in this infinite union iff it's in at least one of the subsets A_i. An element is in an infinite intersection

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots$$

iff it's in each one of the subsets A_i. Note the correspondence of unions to existential quantifiers and intersections to universal quantifiers.

Although our indices i are usually integers, sometimes we'll use some other indexing set I and change our notation slightly

$$\begin{array}{rcl} \bigcup_{i \in I} A_i & = & \{x \in U \,|\, \exists i \in I, x \in A_i\} \\ \bigcap_{i \in I} A_i & = & \{x \in U \,|\, \forall i \in I, x \in A_i\} \end{array}$$

What happens when the indexing set I is empty? The expression $\exists i \in I, x \in A_i$ is always false then, so the empty union is empty. But the expression $\forall i \in I, x \in A_i$ is vacuously true, so the empty intersection is the whole universal set U.