# Math 120 Calculus I

First Test Answers
October 2011

**Scale.** 90–102 A, 80–89 B, 60–79 C, 40–59 D. Median 75.

**1.** [16; 8 points each part] On limits of average rates of change. Let $f(x) = \frac{1}{x}$.

**a.** Write down an expression that gives the average rate of change of this function over the interval $[x, x+h]$, and simplifly the expression.

In general the average rate of change of a function over an interval is the change in $y$ divided by the change in $x$, that is, the length of the interval:

$$\frac{f(x+h) - f(x)}{h.}$$

When the function is $f(x) = \frac{1}{x}$, that becomes

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

which simplifies to

$$\frac{x - (x+h)}{(x+h)xh} = \frac{-h}{(x+h)xh} = \frac{-1}{(x+h)x}$$

**b.** Compute the limit as $h \to 0$ of the average rate of change.

$$\lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

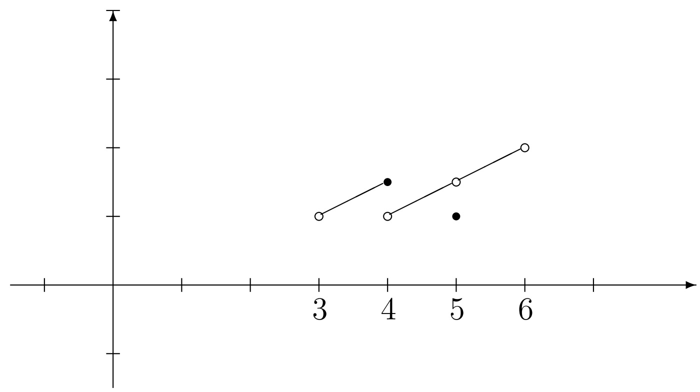
**2.** [15; 5 points for each property] On graphs, limits, and continuity. Draw the graph of *one* function $f$ with domain $(3, 6)$ that has all three of these properties:

**a.** $f$ is continuous on $(3, 6)$ except at $x = 4$ and $x = 5$, but $f(4)$ and $f(5)$ are defined.

**b.** The limit $\lim_{x \to 4} f(x)$ does not exist.

**c.** The limit $\lim_{x \to 5} f(x)$ does exist but $f$ is not continuous at $x = 5$.

Here's an example function which satisfies the conditions.




**3.** [15; 5 points each property] On asymptotes. Draw the graph of *one* function $f$ that illustrates all three of these limits:

**a.** $\lim_{x \to 2^-} f(x) = \infty$.

**b.** $\lim_{x \to 2^+} f(x) = -\infty$.

**c.** $\lim_{x \to \infty} f(x) = 1$.

The graph should have a vertical asymptote at $x = 2$ and a horizontal one at $y = 1$.

As you move right along your curve, as you get close to 2, the curve should reach up toward the top of the verical asymptotote $x = 2$. Then it should come up from the bottom of the vertical asympotote and continue right.

Then as you continue going right it should get close to the horizontal asymptote $y = 1$.

By the way, an example function that has this behavior is $f(x) = \frac{x-3}{x-2}$.

**4.** [40; 8 points each part] Evaluate the following limits. If they diverge to $\pm\infty$ it is enough to say they don't exist.

**a.** $\lim_{x\to\pi} (x^2 + \sqrt{x} - 3\cos x)$

Since the function $x^2+\sqrt{x}-3\cos x$ is continuous, its value at $\pi$ is the limit. $\pi^2 + \sqrt{\pi} - 3\cos\pi$. (If you want, you can simplify it, but remember that $\cos\pi = -1$.)

**b.** $\lim_{x\to 2} \dfrac{x^2+x-6}{x^2-4}$

Both the numerator and the denominator approach 0, so you have to do some analyisis to find the limit. Factor the numerator and denominator.

$$\lim_{x\to 2} \frac{(x+3)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2} \frac{x+3}{x+2} = \frac{5}{4}$$

**c.** $\lim_{x\to\infty} \dfrac{3x^3-2x^2+x}{5x^2+4x+1}$

The numerator has a higher degree than the denominator, so as $x\to\infty$, the quotient $\to\infty$.

**d.** $\lim_{x\to 2} \dfrac{|2-x|}{2-x}$.

The left limit is

$$\lim_{x\to 2^-} \frac{|2-x|}{2-x} = \lim_{x\to 2^-} \frac{2-x}{2-x} = 1$$

while the right limit is

$$\lim_{x\to 2^+} \frac{|2-x|}{2-x} = \lim_{x\to 2^+} \frac{x-2}{2-x} = -1.$$

Since they don't agree, the limit does not exist.

**e.** $\lim_{h\to 0} \dfrac{(x+h)^3-x^3}{h}$

$$\begin{aligned} &= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\ &= \lim_{h\to 0} (3x^2+3xh+h^2) = 3x^2 \end{aligned}$$

**5.** [16] Recall that we define $\lim_{x\to a} f(x) = L$ to mean

$$\forall\epsilon>0, \exists\delta>0, \forall x\,(0<|x-a|<\delta \Rightarrow |f(x)-L|<\epsilon).$$

Here, you will use that definition to prove that if $\lim_{x\to 3} f(x) = 4$, then $\lim_{x\to 3} -f(x) = -4$.

**a.** [4] Use the above definition of limits to translate what the given information, $\lim_{x\to 3} f(x) = 4$, says in terms of $\epsilon$ and $\delta$.

It says

$$\forall\epsilon>0, \exists\delta>0, \forall x\,(0<|x-3|<\delta \Rightarrow |f(x)-4|<\epsilon).$$

If you prefer, you can say it in words. For each positive number $\epsilon$, there is a positive number $\delta$ such that whenever $x$ is within $\delta$ of 3, but not equal to 3, then $f(x)$ is within $\epsilon$ of 4.

**b.** [4] Translate what you're to prove, $\lim_{x\to 3} -f(x) = -4$, in terms of $\epsilon$ and $\delta$.

It says

$$\forall\epsilon>0, \exists\delta>0, \forall x\,(0<|x-3|<\delta \Rightarrow |-f(x)+4|<\epsilon).$$

**c.** [8] Now, let $\epsilon>0$ be given. Explain why there exists a $\delta$ as required in part **b**. Point out where in your argument you use the information given in part **a**.

By part **a** there is a $\delta >$ such that

$$\forall x\,(0<|x-3|<\delta \Rightarrow |f(x)-4|<\epsilon).$$

But the conclusion for part **b**, namely,

$$|-f(x)+4|<\epsilon$$

is equivalent to

$$|f(x)-4|<\epsilon$$

which is the conclusion for part **a**. Therefore, this same $\delta$ works for part **b** as well.