Math 120 Calculus I

First Test Extra Answers

October 2011

6. [16; 8 points each part] On limits of average rates of change. Let $f(x) = 3x^2 + 5x - 2$.

a. Write down an expression that gives the average rate of change of this function over the interval $[x, x+h]$, and simplifly the expression.

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ = & \frac{3(x+h)^2 + 5(x+h) - 2 - (3x^2 + 5x - 2)}{h} \\ = & \frac{3x^2 + 6xh + 3h^2 + 5x + 5h - 2 - 3x^2 - 5x + 2}{h} \\ = & \frac{6xh + 3h^2 + 5h}{h} \\ = & 6x + 3h + 5 \end{aligned}$$

b. Compute the limit as $h \to 0$ of the average rate of change.

$$(6x + 3h + 5) \to (6x + 5)$$

7. On graphs, limits, and continuity. Draw the graph of *one* function f with domain $(0, 5)$ that has all three of these properties:

a. The domain of f is $(0, 5)$.

b. f has a vertical asymptote at $x = 3$.

c. $\lim\limits_{x \to 1} f(x) = +\infty$.

d. f has a jump discontinuity at $x = 2$.

Here's an example

8. Evaluate the following limits. If they diverge to $\pm\infty$ it is enough to say they don't exist.

a. $\lim\limits_{x \to 0} \frac{5 \sin x \cos x}{7x} = \frac{5}{7} \lim\limits_{x \to 0} \frac{\sin x \cos x}{x}$
$= \frac{5}{7} \left(\lim\limits_{x \to 0} \frac{\sin x}{x} \right) \left(\lim\limits_{x \to 0} \cos x \right) = \frac{5}{7}$

b. $\lim\limits_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim\limits_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)}$
$= \lim\limits_{x \to 1} \frac{x^2 + x + 1}{x + 1} = \frac{3}{2}$

c. $\lim\limits_{x \to \infty} \frac{3x^2 - 2x^2 + x}{5x^2 + 4x + 1}$

Both the numerator and denominator are quadratic polynomials. The leading coefficient of the numerator is 1, while the leading coefficient of the denominator is 5. Therefore the limit is $\frac{1}{5}$.

d. $\lim\limits_{x \to 5} (4x \sin \pi x + \sqrt{x^2 - 9})$

The function is continuous, so the limit is $20 \sin 5\pi + \sqrt{5^2 - 9}$. If you want to, you can simplify that to 4.

9. Use the definition of limit to prove that $\lim\limits_{x \to 2} \frac{1}{x} = 0.5$.

Let a positive number ϵ be given. Determine what positive value δ so that whenever $0 < |x - 2| < \delta$ then $\left| \frac{1}{x} - 0.5 \right| < \epsilon$.

You may use either an algebraic approach or a geometric approach or a combination. Explain your steps as you determine what δ will work.

In order to get x^2 between $0.5 - \epsilon$ and $0.5 + \epsilon$, you'll need x between $\frac{1}{0.5 + \epsilon}$ and $\frac{1}{0.5 - \epsilon}$. If you take δ to be the minimum distance between these two numbers and 2, then if x is within δ of 2, then it will be between them both, so that will make x^2 between $0.5 - \epsilon$ and $0.5 + \epsilon$ as required.

Since $\frac{1}{0.5 + \epsilon}$ will be closer to 2, you can actually name δ as $2 - \frac{1}{0.5 + \epsilon}$.