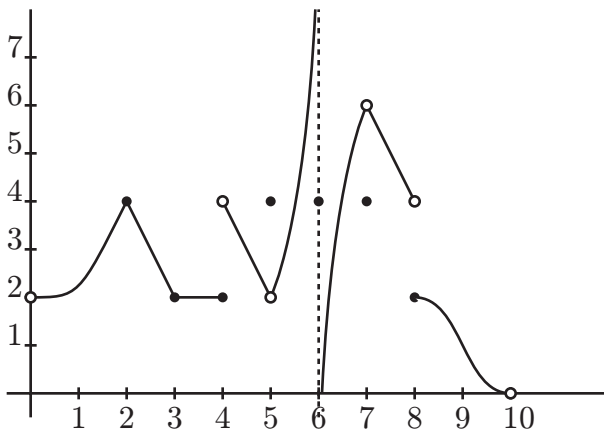




Math 120 Calculus I  
First Test Sample  
October 2011

You may use a calculator. Leave your answers as expressions such as  $e^2 \sqrt{\frac{\sin^2(\pi/6)}{1 + \ln 10}}$  if you like. Show all your work for credit. Points for each problem are in square brackets.

1. [18; 9 points each part] All parts refer to the graph of the function  $f(x)$  with domain  $0 < x < 10$  which is shown below. (All dots have integer coordinates and the dotted line is a vertical asymptote.)



a. List the values of  $a$  in the interval  $(0, 10)$  for which the limit  $\lim_{x \rightarrow a} f(x)$  does not exist.

4 and 8 since the left and right limits aren't equal. 6 because of the asymptote.

b. List the values of  $a$  in the interval  $(0, 10)$  at which  $f$  is not continuous.

Besides 4, 6, and 8, also 5 and 7 since the limits don't equal the values there.

2. [10] Let  $f(x) = x^3$ . Compute the average rate of change of  $f$  over the interval  $[1, 3]$ .

The average rate of change of a function  $f$  over an interval  $[a, b]$  is  $\frac{f(b) - f(a)}{b - a}$ , so for this function

over this interval, it's  $\frac{3^3 - 1^3}{3 - 1}$  (which simplifies to 13).

3. [20; 10 points each part] Recall that we define  $\lim_{x \rightarrow a} f(x) = L$  to mean

$$\forall \epsilon > 0, \exists \delta > 0, \forall x (0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

Here, you will use that definition to prove that  $\lim_{x \rightarrow 3} (10 - 2x) = 4$ .

a. Let  $\epsilon > 0$ . Determine what the value is needed for  $\delta$ .

We need to find  $\delta > 0$  so that if  $0 < |x - 3| < \delta$ , then  $|(10 - 2x) - 4| < \epsilon$ . Simplify the required condition on  $\epsilon$ .  $|6 - 2x| < \epsilon$ , so  $|3 - x| < \frac{\epsilon}{2}$ . Thus,  $\delta = \frac{\epsilon}{2}$  works.

b. In your own words, explain why whenever  $x$  is within your  $\delta$  of 3, then  $f(x)$  will be within  $\epsilon$  of 4.

There are many ways you can explain it. Here's one.

If  $x$  is within  $\frac{\epsilon}{2}$  of 3, then  $2x$  will be within  $\epsilon$  of 6, therefore  $f(x) = 10 - 2x$  will be within  $\epsilon$  of  $10 - 6$ , which is 4, as required.

4. [42; 7 points each part] Evaluate the following limits. If they diverge to  $\pm\infty$  it is enough to say they don't exist.

a.  $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - x - 6}$

Since the numerator and denominator both approach 0, you'll need to do something to find the limit. Factor the numerator and denominator, cancel, and then take the limit.

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - x - 6} &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 1)}{(x - 3)(x + 2)} \\ &= \lim_{x \rightarrow 3} \frac{x + 1}{x + 2} = \frac{4}{5} \end{aligned}$$

b.  $\lim_{x \rightarrow 3} \frac{2x^2 - 2x - 3}{x^2 - x - 6}$

The numerator approaches 9 while the denominator approaches 0, so the limit doesn't exist. (Or you can say it diverges to  $\pm\infty$ .)

$$\mathbf{c.} \lim_{x \rightarrow \infty} \frac{2x^2 - 2x - 3}{x^2 - x - 6}$$

Here, the numerator and denominator both approach  $\infty$ . Both are quadratic polynomials, so the ratio of their leading coefficients gives, the limit, which is 2. (You can divide both the numerator and denominator by  $x^2$  to see it.)

$$\mathbf{d.} \lim_{x \rightarrow 0} \frac{\cos x}{x + \sin x}.$$

This is somewhat like part **b**. Here the numerator approaches 1 while the denominator approaches 0, so the limit of doesn't exist.

$$\mathbf{e.} \lim_{x \rightarrow 0} \frac{|x|}{x}$$

If  $x$  is positive, then  $|x|$  equals  $x$ , so  $\frac{|x|}{x} = 1$ . But if  $x$  is negative, then  $|x|$  equals  $-x$ , so  $\frac{|x|}{x} = -1$ . Therefore, the right limit equals 1, but the left limit equals  $-1$ . Hence, the limit doesn't exist.

$$\mathbf{f.} \lim_{x \rightarrow 0} \frac{3 \sin x}{2x}$$

Since the  $\frac{3}{2}$  is a constant, you can rewrite the limit as  $\frac{3}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x}$ . You'll recognize that the limit  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  equals 1, so the original limit equals  $\frac{3}{2}$ .

**5.** [10] Let  $f(x) = 1 + x^2 - x^5$ . Show that the equation  $f(x) = -3$  has a solution in the interval  $[0, 3]$ . (Hint: IVT.)

Since  $f(0) = 1$  while  $f(3)$  is much less than  $-3$  (it's  $-233$ ), and the function is continuous, therefore by the Intermediate Value Theorem,  $f$  takes on all values between 1 and  $-233$ , in particular, it takes on  $-3$  somewhere in the interval  $[0, 3]$ .