Math 120 Calculus I

Second Test Sample Answers

November 2011

1. [15] Let g be the function inverse to the function $f(x) = x^3 + x$. Determine $g'(2)$. (Note that $g(2) = 1$.)

The inverse function theorem says that the derivative $g'(2)$ is the reciprocal of $f'(1)$. Since $f'(x) = 3x^2 + 1$, therefore $f'(1) = 4$, so $g'(2) = \frac{1}{4}$.

2. [15] Recall the following two limits we proved in class

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1, \quad \text{and} \quad \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

Use those two limits to prove that the derivative of $f(x) = \cos x$ is $-\sin x$. The sum formula for cosines will be useful here: $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\begin{aligned}
& (\cos x)' \\
= & \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
= & \lim_{h\to 0} \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
= & \lim_{h\to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) \\
= & (\cos x) \lim_{h\to 0} \frac{\cos h - 1}{h} - (\sin x) \lim_{h\to 0} \frac{\sin h}{h} \\
= & (\cos x)0 - (\sin x)1 = -\sin x
\end{aligned}$$

3. [40; 8 points each part] Differentiate the following functions. (Do not simplify your answers.)

a. $f(x) = \tan 3x + \arctan 3x$.

$$f'(x) = 3\sec^2 3x + \frac{3}{1 + 9x^2}$$

b. $f(x) = e^x\sqrt{x}$.

$$f'(x) = e^x\sqrt{x} + \frac{e^x}{2\sqrt{x}}.$$

c. $f(x) = \dfrac{1 + \tan x}{x^2 + 5x + 2}$.

$$f(x) = \frac{(\sec^2 x)(x^2 + 5x + 2) - (1 + \tan x)(2x + 5)}{(x^2 + 5x + 2)^2}.$$

d. $f(x) = x^x$. (Suggestion: logarithmic differention)

First, take logs. $\ln f(x) = x \ln x$.

Next, take $\frac{d}{dx}$. $\dfrac{f'(x)}{f(x)} = \ln x + 1$.

Finally solve for f'. $f'(x) = x^x(\ln x + 1)$.

e. $f(x) = \sqrt{\cos 2x}$.

$$f(x) = \frac{-2\sin 2x}{2\sqrt{\cos 2x}}.$$

4. [15] (Page 126, exercises 27-30.) Here are the graphs of four functions f_1, f_2, f_3, and f_4.

Match each of the following four graphs with their derivatives f_1', f_2', f_3', and f_4'.

f_1' is (b). f_2' is (a). f_3' is (d). f_4' is (c)

5. [15] Supppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area $A = 4\pi r^2$. Show that under these circumstances the drop's radius r increases at a constant rate. (Recall that the volume V of a sphere is $V = \frac{4}{3}\pi r^3$.)

We're given that $\frac{dV}{dt}$ is proportional to A, so $\frac{dV}{dt} = kA$ for some constant k.

Differentiate the equation $V = \frac{4}{3}\pi r^3$ with respect to t to find that $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Therefore $kA = 4\pi r^2 \frac{dr}{dt}$. But $A = 4\pi r^2$, so $kA = A\frac{dr}{dt}$. Cancelling A, we conclude $\frac{dr}{dt} = k$.

Thus, the drop's radius increases at a constant rate.