First Test Answers
Math 120 Calculus I
September, 2013

Scale. 90–100 A, 80–89 B, 65–79 C. Median 80.

1. [15] On implicit differentiation. The point $(1,1)$ lies in the curve $x^2-y^2=xy^3-1$. Determine the slope of the line tangent to the curve at that point.

Start by taking the derivative of $x^2-y^2=xy^3-1$ with respect to x. You'll get

$$2x-2y\frac{dy}{dx}=y^3+3xy^2\frac{dy}{dx}.$$

Solve for $\frac{dy}{dx}$ from that equation:

$$\frac{dy}{dx}=\frac{2x-y^3}{3xy^2+2y}$$

That's the slope of the tangent line at a point (x,y) on the curve. When (x,y) is the point $(1,1)$ that is equal to

$$\left.\frac{dy}{dx}\right|_{(1,1)}=\frac{2-1}{(3+2)}=0.2.$$

2. [10] Graphs and derivatives.

a. [5] Sketch the graph $y=f(x)$ of a function for which is continuous everywhere except at $x=3$ and differentiable everywhere except at $x=3$ and $x=4$.

There should be some sort of discontinuity at $x=3$ such as a jump, and at $x=4$ there should be a cusp or a corner in the graph, but smooth everywhere except at those two points.

b. [5] Sketch the graph $y=f(x)$ of a function that is differentiable everywhere such that $f'(3)=0$, $f'(4)=1$ and $f'(5)=0$.

There should be horizontal tangents at $x=3$ and at $x=5$, and the tangent should slope up at $x=4$ with a slope of 1.

3. [15] Recall the definition of derivatives in terms of limits, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Use that definition to show that the derivative of $f(x)=2x^3+x$ is $f'(x)=6x^2+1$. (Do not use any of the rules of differentiation, just the definition.) Here are the steps if you don't leave any of them out. Some can be combined, of course.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(2(x+h)^3+(x+h)-(2x^3+x)}{h}\\
&= \lim_{h\to 0}\frac{2(x^3+3x^2h+3xh^2+h^3)+x+h-2x^3-x}{h}\\
&= \lim_{h\to 0}\frac{6x^2h+6xh^2+2h^3+h}{h}\\
&= \lim_{h\to 0}6x^2+6xh+2h^2+1\\
&= 6x^2+1
\end{aligned}$$

4. [15] On logarithmic differentiation. The function $y=f(x)=x^{\sin x}$ cannot be differentiated by the power rule since the exponent is not constant, and it can't be differentiated by the exponential rule since the base is not constant, but you can find its derivative with logarithmic differentiation. Find its derivative. Show your work, and write carefully. Express your answer $f'(x)$ in terms of x.

First take ln of the equation $y=f(x)=x^{\sin x}$ to get

$$\ln f(x)=(\sin x)\ln x.$$

Then differentiate with respect to x. You'll need the product rule.

$$\frac{f'(x)}{f(x)}=(\cos x)\ln x+(\sin x)/x.$$

Finally solve for $f'(x)$ to get

$$f'(x)=x^{\sin x}\left((\cos x)\ln x+(\sin x)/x\right).$$

5. [45; 9 points each part] Differentiate the following functions. Do not simplify your answers. Use parentheses properly.

a. $f(t) = 3t^5 - \dfrac{4}{t} + 6 + 9t^{2/3}$

Use the power rule to find

$$f'(t) = 15t^4 + \frac{4}{t^2} + 6t^{-1/3}$$

You can also write that as

$$f'(t) = 15t^4 + 4t^{-2} + \frac{6}{\sqrt[3]{t}}$$

b. $g(x) = e^{5x} + \cos 3x$

You'll need the chain rule to find the derivative of each term.

$$g'(x) = 5e^{5x} - 3\sin 3x$$

c. $y = \dfrac{3 + 4\sqrt{x}}{5 - \tan x}$

Use the quotient rule

$$y' = \frac{(2/\sqrt{x})(5 - \tan x) - (3 + 4\sqrt{x})(-\sec^2 x)}{(5 - \tan x)^2}$$

d. $f(x) = x \arcsin x$. (Note that the inverse sine function $\arcsin x$ is often written $\sin^{-1} x$, but it does not equal $(\sin x)^{-1}$.)

Use the product rule.

$$f'(x) = \arcsin x + \frac{x}{\sqrt{1 - x^2}}$$

e. $f(\theta) = \theta^2 \sin 3\theta$

Use the product rule and then the chain rule.

$$\begin{aligned} f'(\theta) &= (\theta^2)' \sin 3\theta + \theta^2 (\sin 3\theta)' \\ &= 2\theta \sin 3\theta + \theta^2 (3\cos 3\theta) \end{aligned}$$