Failures to have limits
Math 120 Calculus I
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We're primarily interested in limits that finding limits that exist. After all, every derivative is a limit. But to understand a concept in mathematics, it's important to know its boundaries, and for this concept, it's important to know some examples of limits that don't exist. That is, as x approaches a, $f(x)$ does not approach anything. When that happens, we'll say that the $\lim_{x\to a} f(x)$ does not exist.

One imporant way that limits don't exist is that they go off to infinity. For example, $\displaystyle\lim_{x\to 3} \frac{1}{(x-3)^2}$. As when x is close to 3, either slightly greater than 3 or slightly less than 3, the denominator $(x-3)^2$ is a very small positive number, so it's reciprocal is a very large positive number. We'll write $\displaystyle\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$ and say that the limit diverges to ∞. You can see that's what happens since the graph $y = f(x)$ is asymptotic to the vertical line $x = 3$ getting closer nearer the top of the line.

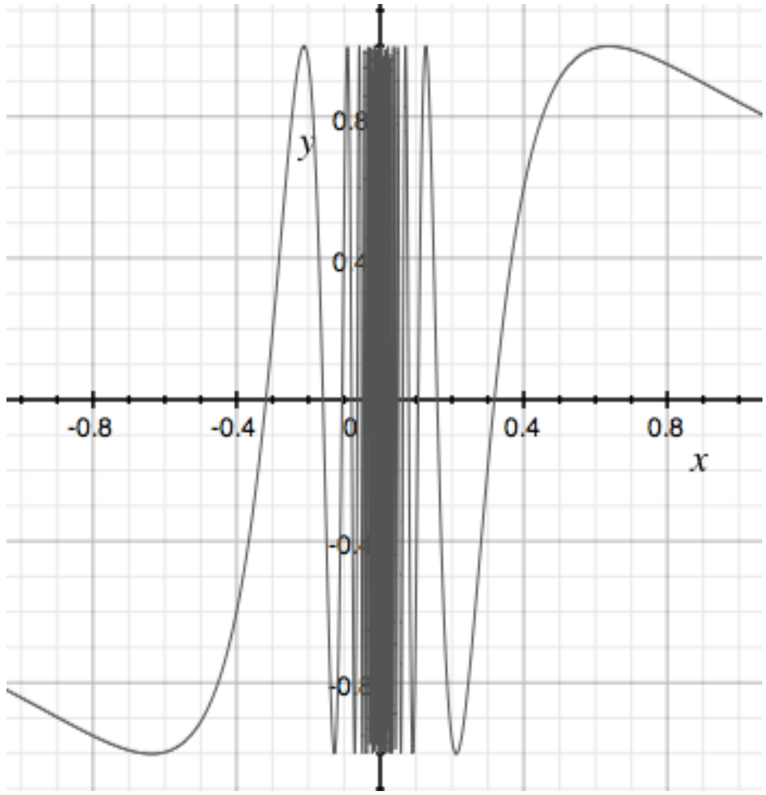
A similar example is $\displaystyle\lim_{x\to 3} \frac{1}{x-3}$. When x is slightly greater than 3, the denominator $x-3$ is a very small positive number, so its reciprocal is a very large positive number. But when x is slightly less than 3, the denominator $x-3$ is slightly less than 0, so its reciprocal is near $-\infty$. We'll write $\displaystyle\lim_{x\to 3} \frac{1}{x-3} = \pm\infty$ and say that the limit diverges to $\pm\infty$. The graph of this function is also asymptotic to the vertical line $x = 3$, but this time on the left it's near the bottom of the line but on the right it's near the top of the line.

Some times there's a jump at $x = a$. That doesn't happen when the function is given by a single expression, but it can happen when the function is defined by cases. For example, if we define f by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 3 \\ 2x & \text{if} \quad x \geq 3 \end{cases}$$

then there is a jump in the graph $y = f(x)$ at $x = 3$. As x approaches 3 from the left, $f(x) = x^2$ approaches 9. But as x approaches 3 from the right, $f(x) = 2x$ approaches 6. We can say the "left limit" is 9 while the "right limit" is 6. Since the number you get depends on the direction you're approaching 3, the limit doesn't exist.

There are other ways that the limit might not exist. Consider the function $f(x) = \sin\frac{1}{x}$. The limit $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. Imagine what happens as you let x approach 0 from the right. Its reciprocal $1/x$ approaches $+\infty$. As that happens the sine of it goes through infinitly many cycles, 0 to 1 to 0 to -1 to 0. That means $y = \sin\frac{1}{x}$ oscillates between -1 and 1 infinitely many times; it is not approaching any particular number.



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