# What are limits?
Math 120 Calculus I

D Joyce, Fall 2011

**We need limits!** As we've seen, we need limits to define derivatives. We will define the derivative $f'(x)$ of a function $f$ at $x$ by one of the two equivalent limits

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x}.$$

It's time now to formally define limits. Note that in this limit, in fact, in all limits, the variable, $h$ in the first limit and $b$ in the second, that's approaching the constant value, 0 in the first case and $x$ in the second, is never taken to be equal to the constant.

Not all the limits we will use are of this form that we'll use in the definition of derivatives. We'll have other uses for limits, too. So, let's change the notation a little to make our definition of limit a general one. We'll define what the expression

$$\lim_{x\to a} f(x) = L$$

means. We'll also express this limit as follows: as $x \to a$, $f(x) \to L$. We'll assume that the function $f$ is defined near $a$, but we won't require that $f$ be defined at $a$, in fact, for our application of derivatives, it never will be.

**The formal definition.** We'll look at the formal definition and analyze it to see what it means. It took mathematicians about 200 years to come up with this definition, from the time that Fermat started to use limits informally to when Cauchy and Weierstrass came up with the definition.

You would think that the definition would say that $\lim_{x\to a} f(x) = L$ means something like "if $x$ is close to $a$, then $f(x)$ is close to $L$," but that doesn't work. You need something backwards instead. The formal definition is closer to "you can make $f(x)$ arbitrarily close to $L$ by insisting that $x$ be sufficiently close to $a$."

*Definition.* Assume that a function $f$ is defined near $a$ (but perhaps not defined at $a$). We'll say that $f(x)$ approaches a limit $L$ as $x$ approaches $a$ when it is the case that for each positive value $\epsilon$, there is a positive value $\delta$ (which may depend on $\epsilon$) such that whenever

$$0 < |x-a| < \delta$$

it is the case that

$$|f(x)-L| < \epsilon.$$

That's the definition. The last part is best interpreted as a statement about distances. It says that whenever $x$ is not equal to $a$ but within $\delta$ of $a$, then $f(x)$ is within $\epsilon$ of the limit $L$.

The first part of the definition contains two quantifiers. A quantifier introduces a new variable and indicates the part that variable plays. There are two kinds of quantifiers—universal quantifiers and existential quantifiers—and both appear in this definition. The phrase "for each positive value of $\epsilon$" is a universal quantifier. It indicates that the rest of the statement has to be true for *all* positive $\epsilon$. Universal quantifiers are very common. Every time the word "every", "all", or "each" occurs in English, there's a universal quantifier. There is a special mathematical symbol that is often used to abbreviate universal quantifiers, an upside down A. Here, it would be denoted $\forall\epsilon > 0$.

The phrase "there is a positive value $\delta$" is an existential quantifier. It indicates that there is at least one positive value for $\delta$ that makes the remaining statment true. Existential quantifiers are also very common. Every time "there is", "there exists", "some", "a", or "an" occurs in English, there's an extential quantifier. The symbol used to abbreviate existential quantifiers is an upside down E. $\exists\delta > 0$.

One of the things that makes this definition of limit difficult to understand is the pair of quantifiers. $\forall\epsilon > 0, \exists\delta > 0$. The existential quantifer comes after the universal one, and that means the value of $\delta$ can depend on the value of $\epsilon$. There's actually a third quantifier in the definition. The "whenever" is a universal quantifier for the $x$ that appears in the rest of the definition.

Some adverbs can also indicate quantifiers. When we say there is a $\delta > 0$ so that whenever $x$ is within $\delta$ of $a$, we could also say that whenever $x$ is sufficiently close to $a$. Likewise when we say that for all $\epsilon > 0$ that $f(x)$ is within $\epsilon$ of $L$, we could say that $f(x)$ can be made arbitrarily close to $L$. The words "sufficiently" and "arbitrarily" indicate existential quantifiers and universal quantifiers but don't explicitly mention variables. Using them, we can state the definition in a way that sounds better in English as follows

> $\lim_{x\to a} f(x) = L$ means that by taking $x$ sufficiently close to $a$ (but not equal to $a$), we can make $f(x)$ arbitrarily close to $L$.

We could express this condition more symbolically as

$$\forall\epsilon > 0, \exists\delta > 0, \forall x\,(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

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