Alternating Series and Absolute Convergence
Math 122 Calculus III

D Joyce, Fall 2012

For the most part we've looked at positive series, those series with positive terms. We've also considered geometric series with negative ratios r. For those geometric series, the signs of the terms alternate between positive and negative.

We'll consider now series that have both positive and negative terms. Many of the most useful ones are *alternating series* whose terms have alternating signs.

We can't use most of the tests for convergence we've considered already since they're for positive terms. The only one that still applies is the Term Test which says that if the terms of a series don't approach 0, then the series diverges, that is to say, for a series to converge, its terms must approach 0.

Absolutely convergent series. Some series have terms that are so close to zero that it doesn't matter whether you add or subtract them, the series will converge to something no matter what. These are called absolutely convergent series.

Theorem 1. Given a series $\sum a_n$, if the series of absolute values of its terms converges, that is, if $\sum |a_n|$ converges, then so does the original series. When that's the case, we say the original series *absolutely converges*

Proof. Suppose $\sum |a_n|$ converges. We'll treat the series $\sum a_n$ as a sum of a positive series $\sum b_n$ and a negative series $\sum c_n$. Here's how. If a term a_n is positive, let $b_n = a_n$ and $c_n = 0$, but if a_n is negative, then let $b_n = 0$ and $c_n = a_n$. Then $a_n = b_n + c_n$. Now $b_n \leq |a_n|$, so the series $\sum |a_n|$ dominates the series $\sum b_n$, so by the comparison test $\sum b_n$ also converges. Likewise $\sum |a_n|$ dominates $\sum(-c_n)$, so $\sum(-c_n)$ converges, and so does $\sum c_n$. Since $\sum a_n$ is the sum of two convergent series, it, too, converges. Q.E.D.

For example, the series $\sum \dfrac{1}{n^2}$ converges, so the alternating series $\sum \dfrac{(-1)^n}{n^2}$ absolutely converges.

For another example, the series $\sum \dfrac{\sin n}{n^2}$ absolutely converges. Why? Because the series of absolute values of its terms, $\sum \dfrac{|\sin n|}{n^2}$ converges because it's dominated by the convergent series $\sum \dfrac{1}{n^2}$. Note that $\sum \dfrac{\sin n}{n^2}$ is not an alternating series, but some of its terms are positive and some are negative.

Conditionally convergent series. There are some series that converge, but aren't absolutely convergent. We'll call them *conditionally convergent* series. We'll see that an example of this is the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots .$$

It's not absolutely convergent since the series of the absolute values of its terms is the harmonic series which we know diverges. In the next paragraph, we'll have a test, the Alternating Series Test, which implies that this alternating harmonic series converges. In fact, the sum of this series is $\ln 2$, but we won't show that until we look at power series.

When you have a conditionally convergent series, Riemann's theorem says you can rearrange it's terms so that it converges to a different number, in fact, to any different number. We won't prove Riemann's theorem in this course, but in a moment we'll see how you can rearrange the terms of the alternating harmonic series to get it to sum to a different number than $\ln 2$. Absolutely convergent series don't behave like that; no matter how you rearrange the terms of an absolutely convergent series, it always converges to the same number.

See how this compuatation involves rearranging the terms of the alternating harmonic series.

$$\begin{aligned}
\ln 2 &= 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}+\cdots \\
&= 1-\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{8}+\tfrac{1}{5}-\tfrac{1}{10}-\cdots \\
&= (1-\tfrac{1}{2})-\tfrac{1}{4}+(\tfrac{1}{3}-\tfrac{1}{6})-\tfrac{1}{8}+(\tfrac{1}{5}-\tfrac{1}{10})-\cdots \\
&= \tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{8}+\tfrac{1}{10}-\cdots \\
&= \tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\cdots) \\
&= \tfrac{1}{2}\ln 2
\end{aligned}$$

Which is absurd since $0 \neq \ln 2$.

Leibniz' Alternating Series Test. This is a test which we'll use to show lots of alternating series converge.

Theorem 2 (Leibniz)**.** If the absolute values of the terms of an alternating series converge monotonically to 0, then the series converges.

Proof. Let $\displaystyle\sum_{k=0}^{\infty}(-1)^k a_k$ be an alternating series where the absolute values of the terms a_k form a sequence that decreases to 0. (It's actually enough that the sequence is nonincreasing and its limit is 0.)

The even partial sums $S_0, S_2, S_4, \ldots$ form a decreasing sequence since $S_2 = S_0 - (a_1 - a_2) < S_0$, $S_4 = S_2 - (a_3 - a_4) < S_2$, etc. Likewise, the odd partial sums $S_1, S_3, S_5, \ldots$ form an increasing sequence since $S_3 = S_1 + (a_2 - a_3) > S_1$, $S_5 = S_3 + (a_4 - a_5) > S_3$, etc. Note that all the odd partial sums are less than all the even partial sums. The sequence of even partial sums is decreasing and bounded below, so it has a limit S_{e}, while the sequnece of odd partial sums is increasing and bounded above, so it also has a limit S_{o}.

As the difference between the even partial sum S_{2n} and the previous odd partial sum S_{2n-1} is a_{2n}, and both S_{e} and S_{o} lie between S_{2n-1} and S_{2n}, therefore the difference $S_{\mathrm{e}} - S_{\mathrm{o}}$ is less than S_{2n}. But $S_{2n} \to 0$. Therefore, $S_{\mathrm{e}} = S_{\mathrm{o}}$. Thus, that number is the limit of the sequence of all the partial sums. Hence, the series converges. Q.E.D.

As an example of its use, Leibniz' alternating series test implies that the alternating harmonic series converges.

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