CLARK UNIVERSITY 1887

L'Hôpital's Rule
Math 122 Calculus III
D Joyce, Fall 2012

L'Hôpital's rule is a way of determining the limits for two indeterminant forms, namely $\frac{0}{0}$ and $\frac{\infty}{\infty}$, and other indeterminant forms which can be converted into those forms such as $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0.

This rule was in L'Hôpital's 1696 textbook on calculus, the first textbook on calculus ever published. L'Hôpital learned it from Johann Bernoulli.

Indeterminant forms. Frequently you can tell what value an expression is approaching by examining what its components are approaching. For example, for the limit, $\lim_{x \to 3} \dfrac{x^2-4}{x^2-3x+2}$, the numerator is approaching 5 while the denominator is approaching 2, so, of course, the limit is $\frac{5}{2}$. This $\frac{5}{2}$ is a *determinant form*; whenever the numerator approaches 5 and the denominator approaches 2, the quotient approaches $\frac{5}{2}$.

But the limit

$$\lim_{x \to 2} \frac{x^2-4}{x^2-3x+2}$$

is different. Both the numerator and denominator approach 0. This limit has the indeterminant form $\frac{0}{0}$. In order to determine the limit, you'll have to look at more than just its form. For this limit, it's easy; just factor the numerator and denominator and cancel the common factor $x-2$ to see that the limit is

$$\lim_{x \to 2} \frac{x+2}{x-1} = 4.$$

We need to examine limits that have indeterminant forms to see what they are; the forms themselves don't tell us. Other indeterminant forms are mentioned in the introductory paragraph.

L'Hôpital's rule for $\frac{0}{0}$. The limit of this indeterminant form depends on the rates that the numerator and denominator approach 0. If the numerator approaches 0 faster than the denominator, then the limit will be small; if slower, large. Rates are derivatives, and that suggests that we can replace the numerator and denominator by their derivatives. That's L'Hôpital's rule. Here's its statement.

Theorem 1 (L'Hôpital's rule for $\frac{0}{0}$)**.** When considering a limit of $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ where both the numerator $f(x) \to 0$ and the denominator $g(x) \to 0$, if the denominator $g(x) \neq 0$ near a and if $\lim_{x \to a} \dfrac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = L$, also.

This is valid even when $a = \pm\infty$.

We'll give a geometric proof of the special case when $g'(x) \to D \neq 0$ in class. The general case relies on something called Cauchy's mean-value theorem, and we'll skip the general case.

For an example of it's use, consider the limit we looked at before, $\lim_{x \to 2} \dfrac{x^2-4}{x^2-3x+2}$. Both the numerator and denominator approach 0 as $x \to 2$. The derivative of the numerator is $2x$ while the derivative of the denominator is $2x-3$. We know the quotient of these derivatives approaches 1, $\lim_{x \to 2} \dfrac{2x}{2x-3} = 4$. Therefore the quotient of the original functions does, too. $\lim_{x \to 2} \dfrac{x^2-4}{x^2-3x+2} = 4$.

Here's another example.

$$\lim_{x \to 0} \frac{x^2}{\sin x} \overset{\text{L'H}}{=} \lim_{x \to 0} \frac{2x}{\cos x} = \frac{0}{1} = 0.$$

Sometimes you'll use L'Hôpital's rule twice.

$$\begin{aligned} \lim_{x \to 0} \frac{e^x - x - 1}{x^2} &\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{e^x - 1}{2x} \\ &\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2} \end{aligned}$$

L'Hôpital's rule for $\frac{\infty}{\infty}$. The statement is almost identical to the $\frac{0}{0}$ statement.

Theorem 2. When considering a limit of $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}$ where both the numerator $f(x)\to\infty$ and the denominator $g(x)\to\infty$, if the denominator $g(x)\neq 0$ near a and if $\displaystyle\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$, then $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=L$, also.

Here's an outline of a proof. There are some details missing.

Let's restate the $\frac{0}{0}$ form as follows. If $f\to 0$ and $g\to 0$, then $\dfrac{f}{g}\dfrac{g'}{f'}\to 1$.

Suppose now that $f\to\infty$ and $g\to\infty$. Then $1/f\to 0$ and $1/g\to 0$. Apply the $\frac{0}{0}$ form to $1/f$ and $1/g$ to get $\dfrac{(1/f)}{(1/g)}\dfrac{(1/g)'}{(1/f)'}\to 1$. That says

$$\frac{g}{f}\,\frac{-g'/g^2}{-f'/f^2}\to 1.$$

That simplifies to

$$\frac{f}{g}\,\frac{g'}{f'}\to 1$$

which is what we wanted to show.

For an example, take

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}\overset{\text{L'H}}{=}\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

That example shows that the logarithm function grows to infinity much slower than the square root function. You can generalize that example to show that logs grow to infinity much slower than any positive power.

Likewise, you can show that e^x grows to infinity much faster than any power, that is $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$ for any constant n.

Math 122 Home Page at
`http://math.clarku.edu/~djoyce/ma122/`