CLARK UNIVERSITY 1887

Summary of definitions and theorems for sequences
Math 122 Calculus III

D Joyce, Fall 2012

This is a summary of the main points we've discussed in class about the infinite sequences. We'll use them soon in our study of series.

Definition 1. An infinite *sequence* of real numbers is simply a function $\mathbf{N} \to \infty$, written

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

or $\{a_n\}_{n=1}^{\infty}$.

When we have a particular sequence under consideration, we'll either write it out as a list, such as $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$, or specify a general term as a formula, here it would be $a_n = 1/n^2$.

Definition 2 (Monotonicity)**.** A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *increasing* or *strictly increasing* if for each n, $a_n < a_{n+1}$. It's *nondecreasing* or *weakly increasing* if for each n, $a_n \leq a_{n+1}$. *Decreasing* sequences are defined analogously. A sequence that satisfies any of these conditions is said to be *monotone.*

Definition 3 (Boundedness)**.** A sequence is *bounded* if it is bounded as a set. More precisely $\{a_n\}_{n=1}^{\infty}$ is *bounded below* by B if $B \leq a_n$ for all n, and it's *bounded above* by U if $a_n \leq U$ for all n. If it's bounded both below and above, it's said to be *bounded.*

Definition 4 (Convergence and limits of sequences)**.** A sequence is said to *converge* to the *limit* L, if for each $\epsilon > 0$, there is some N such that beyond the N^{th} term, every term is within ϵ of L. When the sequence is $\{a_n\}_{n=1}^{\infty}$, we'll write $\lim_{n\to\infty} a_n = L$, or more briefly $a_n \to L$. Symbolically, convergence says

$$\forall \epsilon > 0, \exists N, \forall n \geq N, |a_n - L| < \epsilon.$$

A sequence that doesn't converge is said to *diverge.*

Note that limits of sequences are called *discrete* limits since the subscript n only takes integers as values, and the integers are separated from each other, that is, they're discrete.

Theorem 5. A sequence can have at most one limit.

Theorem 6. A nondecreasing sequence bounded above converges to its least upper bound. A nonincreasing sequnce bounded below converges to its greatest lower bound.

Theorem 7. If two sequences have limits, $a_n \to A$ and $b_n \to B$, then their sum $a_n + b_n \to A + B$, their difference $a_n - b_n \to A - B$, and their product $a_n b_n \to AB$. Also, their quotient $a_n/b_n \to A/B$ provided $B \neq 0$.

Theorem 8. If $\lim_{x\to\infty} f(x) = L$, and $a_n = f(n)$, then $a_n \to L$, also

Theorem 9. If $a_n \to L$, and if f is a continuous function, then $f(a_n) \to f(L)$.

Theorem 10 (Pinching theorem)**.** If $a_n \leq b_n \leq c_n$ for all n, and both $a_n \to L$ and $c_n \to L$, then $b_n \to L$, also.

Math 122 Home Page at `http://math.clarku.edu/~djoyce/ma122/`