CLARK UNIVERSITY 1887

Natural numbers
Math 122 Calculus III
D Joyce, Fall 2012

We'll have occasion to distinguish between different kinds of numbers. We'll consider the natural numbers $\mathbf{N}$, the integers $\mathbf{Z}$, the rational numbers $\mathbf{Q}$, the real numbers $\mathbf{R}$, and the complex numbers $\mathbf{C}$.

The natural numbers, N, and mathematical induction. These are also called positive integers, or whole positive numbers. They include 1, 2, 3, and so forth. We'll use the notation $\mathbf{N}$ for the set of all natural numbers.

A serious study of the foundations of mathematics would study $\mathbf{N}$ in detail. We won't do that, but we will look at one principle that is used to prove statements about all natural numbers. It's called *mathematical induction*. We won't dwell on it, but we'll use it once in a while to prove general statements about $\mathbf{N}$.

In order to use this principle to prove that a statement $S(n)$ is true for all natural numbers n, first prove that $S(1)$ is true. That's called the *base case*. Next prove that for each natural number n, if $S(n)$ is true, then $S(n+1)$ is true. That's called the *inductive step*. If you've done the base case and inductive step, then that's enough to conclude that $S(n)$ is true for all n.

Example 1 (Sum of a finite geometric series)**.** A geometric series is an expression where each term is some constant r times the previous term. The constant r is called the *ratio* of the geometric series. The series if finite if it has a finite number of terms. We'll soon study infinite series, and the first ones we'll study will be infinite geometric series.

If we denote by a the first term of a finite geometric series, and if the series has n terms, then the series is

$$a + ar + ar^2 + \cdots + ar^{n-1}.$$

You've probably already seen a good argument that the sum of this series is $a\,\dfrac{1-r^n}{1-r}$, but let's prove that formally using the principle of mathematical induction.

For the base case, we need to show that the sum of a geometric series with $n = 1$ term is $a\,\dfrac{1-r^1}{1-r}$. But the the series with one term is just a and the purported sum is also a. So the base case has been proved. Usually the base case is obviously true.

Now for the inductive step. We'll assume the statement $S(n)$ is true for n, and we'll show $S(n+1)$ is also true. Here's what they say for the sum of a finite geometric series

$$S(n): \qquad a + ar + ar^2 + \cdots + ar^{n-1} = a\,\frac{1-r^n}{1-r}.$$

$$S(n+1): \qquad a + ar + ar^2 + \cdots + ar^{n} = a\,\frac{1-r^{n+1}}{1-r}.$$

The second statement follows from the first as seen by these equations

$$\begin{aligned} a + ar + ar^2 + \cdots + ar^n &= (a + ar + ar^2 + \cdots + ar^{n-1}) + ar^n \\ &= a\,\frac{1-r^n}{1-r} + ar^n \\ &= a\,\frac{1-r^n+r^n-r^{n+1}}{1-r} \\ &= a\,\frac{1-r^{n+1}}{1-r} \end{aligned}$$

That concludes the inductive step. We've shown that each natural number n, if $S(n)$ is true, then $S(n+1)$ is true.

Since we proved both the base case and the inductive step, we conclude that the statement is true, that is, for each n, $S(n)$ is true. Q.E.D.

We'll look at a few more examples that follow this form to get a better idea of how mathematical induction works.

Example 2. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$.

Example 3. $(2n)! < 2^{2n}(n!)^2$.

Exercise 1. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Exercise 2. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

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