CLARK UNIVERSITY 1887

Survey of Series and Sequences
Math 122 Calculus III

D Joyce, Fall 2012

The goal. One purpose of our study of series and sequences is to understand power series. A power series is like a polynomial of infinite degree. For example,

$$1 + x + x^2 + \cdots + x^n + \cdots$$

is a power series. We'll look at this one in a moment.

Power series have a lot of properties that polynomials have, and that makes them easy to work with. Also, they're general enough to represent lots of important functions like e^x, $\ln x$, $\sin x$, and $\cos x$.

We'll see, for instance, that the function $\dfrac{1}{1-x}$ is represented by a power series for x inside the interval $(-1, 1)$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } x \in (-1, 1).$$

In order to make that statement, we'll have to define just what it means for a series to have a sum, and that will take us a while.

Beyond that, we'll need the theory of Taylor series to develop the following power series for the three important functions e^x, $\cos x$, and $\sin x$. For all x,

$$\begin{array}{rcccccccccccccc}
e^x & = & 1 & + & x & + & \frac{x^2}{2!} & + & \frac{x^3}{3!} & + & \frac{x^4}{4!} & + & \frac{x^5}{5!} & + & \cdots \\
\cos x & = & 1 & & & - & \frac{x^2}{2!} & & & + & \frac{x^4}{4!} & & & - & \cdots \\
\sin x & = & & & x & & & - & \frac{x^3}{3!} & & & + & \frac{x^5}{5!} & & \cdots .
\end{array}$$

Note that the terms of the $\cos x$ series are the even terms of the e^x series, but the $\cos x$ series alternates sign, and the terms of the $\sin x$ series are the odd terms of the e^x series, and, again, the $\sin x$ series alternates sign.

The symbol $n!$ is read "n factorial" or "factorial n" (except by some people who like to say "n shriek" or "n bang"), and it means the product of the integers from 1 through n.

$$n(n-1)(n-2)\cdots 2 \cdot 1.$$

Also, 0! is defined to be equal to 1. We'll have lots of use for factorials while studying Taylor series. Incidentally, you may have come across factorials before if you studied permutations and combinations.

The foundations. We won't look at power series at first; we'll look at series without variables. The term "series" is used to describe an infinite sum.

Definition 1 (Series)**.** A series is a formal expression for an infinite sum. A general series is of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

where the terms $a_1, a_2, a_3 \ldots, a_n, \ldots$ are numbers.

Sometimes we'll use summation notation to describe a series. In that notation, the series in the definition is written $\displaystyle\sum_{n=1}^{\infty} a_n$.

We'll follow two examples as we develop this theory.

Example 2 (A geometric series)**.**

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \tfrac{1}{2^n} + \cdots +$$

The first term is $a_1 = \frac{1}{2}$, the second is $a_2 = \frac{1}{4}$, and the n^{th} term is $\frac{1}{2^n}$. We're interested in the sum of this series, but we'll have to define what the sum of a series is first.

This series is called a geometric series because its terms are in a geometric progression (also called a geometric sequence). In a geometric progression each term is found by multiplying the preceding term by a fixed constant, called the *ratio*. In this example, the ratio is $\frac{1}{2}$.

Example 3 (A harmonic series)**.**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n} + \cdots +$$

In form, this is very similar to the preceding series. We're interested in its sum, too.

This series is called a harmonic series because its terms are in a harmonic progression. The terms in a harmonic progression are reciprocals of the terms in an arithmetic progression. For an arithmetic progression each term is found by adding a fixed constant to the preceding term.

The way that we'll get at the sum of a series is by its partial sums. A partial sum is the sum of finitely many terms at the beginning of the series.

Definition 4 (Partial sums)**.** The n^{th} partial sum, S_n of a series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

Thus, $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, and so forth.

The idea is that the sum of the whole series is the limit of the partial sums. That is, if you keep adding more terms of the series, you'll get close to the sum of the series. But that requires that we define what the limit of sequence of partial sums is. So, we'll formally define what a sequence is, and what its limit is.

Definition 5 (Sequence)**.** A sequence is an infinite list of numbers. A general sequence is of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

Associated to each series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ there are two sequences. First, the terms of the series form a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$. Second, the partial sums $S_1, S_2, S_3, \ldots, S_n, \ldots$ form a sequence. Let's look at the sequence of partial sums for the two examples above.

For the geometric series example, $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, and $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$. Thus, the sequence of partial sums is

$$\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \ldots, \tfrac{2^n-1}{2^n}, \ldots.$$

The n^{th} partial sum is $S_n = \frac{2^n-1}{2^n} = 1 - \frac{1}{2^n}$.

For the harmonic series example, $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, $S_3 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$, and $S_4 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$. It's much harder to find an expression for the n^{th} partial sum.

Limits of sequences and sums of series We're interested in sequences because the limit of the sequence of partial sums of a series will be defined as the sum of the series. So, we want to know what the limit of sequence is and even if the sequence has a limit. Here's the formal definition.

Definition 6 (Limit of a sequence)**.** A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ has a limit L if for each $\epsilon > 0$, there exists a number N such that for all $n \geq N$,

$$|a_n - L| < \epsilon.$$

If the sequence has a limit, we say that sequence *converges*. If it has no limit, we say that it *diverges*. We'll use two notations for the limit of a sequence. One is $\lim\limits_{n\to\infty} a_n = L$. A more abbreviated notation is simply $a_n \to L$.

What this means is that you can make sure that the terms of the sequence are arbitrarily small by going far enough out in the sequence. If you want the terms to be within $\epsilon = 0.0001$ of L, you may need to go as far out as N in the sequence, but if you want to be within $\epsilon = 0.0000001$ of L, your N will have to be much larger as the terms may not be that close to L until much later in the sequence.

The limits we're looking at now, $\lim\limits_{n\to\infty} a_n$, are "discrete" limits, whereas the limits we looked at before of functions, $\lim\limits_{x\to\infty} f(x)$, were "continuous" limits. The main difference is that n only takes integer values, values that are separated from each other, while x takes all real values, and so x varies continuously.

Properties of limits of sequences. Since these discrete limits are defined similarly to continuous limits, they have many of the same properties. Here are a few listed without proof, and one that needs a proof

$$\begin{aligned}
\lim_{n\to\infty} c &= c \quad \text{where } c \text{ is a constant, that is, } a_n = c \text{ all } n \\
\lim_{n\to\infty} (ca_n) &= c \lim_{n\to\infty} a_n \quad \text{where } c \text{ is a constant} \\
\lim_{n\to\infty} (a_n \pm b_n) &= \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \\
\lim_{n\to\infty} (a_n b_n) &= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} a_n \Big/ \lim_{n\to\infty} b_n \quad \text{if the denominator doesn't approach 0} \\
\lim_{n\to\infty} \frac{1}{n} &= 0
\end{aligned}$$

Here's the proof of the last limit. According to our definition, $\lim_{n\to\infty} \dfrac{1}{n} = 0$ means for each $\epsilon > 0$, there exists a number N such that for all $n \geq N$, $|a_n - L| < \epsilon$, that is, $|\frac{1}{n} - 0| < \epsilon$. In order to prove that, let ϵ be positive. We need to find N such that for $n \geq N$, $1/n < \epsilon$. Of course, for $n \geq N$, $1/n \leq 1/N$, therefore we only need to find a positive integer N so that $1/N < \epsilon$. But that's just the Archimedean property of the real numbers. Thus, the last limit follows from the Archimedean property of the real numbers.

There are a couple more important properties of discrete limits.

If two sequences both have limits and each term of the first is less than or equal to the corresponding term of the second, then the limit of the first is less than or equal to the limit of the second. Symbolically, if $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, and for each n, $a_n \leq b_n$, then $L \leq M$.

The pinching lemma, also called the sandwich theorem, says that if two sequences have the same limit, then any intermediate sequence also has the same limit. Symbolically, if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, and for each n, $a_n \leq b_n \leq c_n$, then $\lim_{n\to\infty} b_n = L$.

There's also a property that relates discrete limits to continuous limits. If the terms of a sequence are values of a function and if the continuous limit exists, then so does the discrete limit, and it equals the continuous limit. Symbolically, if for each n, $a_n = f(n)$, and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Example 7 (Sequence of partial sums)**.** Let's look at the sequence of partial sums of the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \cdots$. Its partial sums are $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, \frac{n-1}{n}, \ldots$. The n^{th} term is $\frac{n-1}{n}$, and they get closer to 1 after that. We'll show that the limit L of this sequence is 1. Given $\epsilon > 0$ we need to find out how far, N, we have to go out in the sequence to make sure that the terms beyond N are within ϵ of $L = 1$. Now, the condition $|\frac{2^n-1}{2^n} - 1| < \epsilon$ is equivalent to $\frac{1}{2^n} < \epsilon$, which, in turn, is equivalent to the condition $\frac{1}{\epsilon} < 2^n$, and that's equivalent to $\log_2 \frac{1}{\epsilon} < n$. Thus, if we choose N to be any integer greater than or equal to $\log_2 \frac{1}{\epsilon}$, then the terms beyond N will be within ϵ of 1. Therefore, the limit of this sequence is 1.

Definition 8 (Sum of a series)**.** A series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ has a sum S if the limit of the partial sums is S, $\lim_{n\to\infty} S_n = S$. If the series has a sum, we say that sequence *converges*. If it has no sum, we say that it *diverges*.

Back to the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots + \frac{1}{2^n} \cdots$. Since its partial sums approach the limit 1, therefore the sum of this geometric series is 1. We'll look at general geometric series after the next example.

Example 9 (Divergence of a harmonic series)**.** Consider again the harmonic series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \cdots$. We'll show that it diverges to infinity by showing that its partial sums diverge to infinity. Group the terms together as shown

$$\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) + \cdots$$

The first partial sum is $S_1 = \frac{1}{2}$. The third and seventh partial sums are

$$\begin{aligned}
S_3 &= \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) \\
&\geq \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{2}{2} \\
S_7 &= \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\
&\geq \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{3}{2}
\end{aligned}$$

In general, the $(2^n - 1)^{\text{st}}$ partial sum is

$$S_{2^n-1} = \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \cdots + \left(\tfrac{1}{2^{n-1}+1} + \cdots + \tfrac{1}{2^n}\right) \geq \tfrac{1}{2} + \cdots + \tfrac{1}{2} = \tfrac{n}{2}$$

Since the partial sums grow by at least $\frac{1}{2}$ every time another grouping of terms is added, therefore they diverge to infinity. Thus, this harmonic series is divergent.

Historically, this is an important example. In the 1300s, geometric series were known to converge, and a few others, too, but this was the first known series whose terms approach 0 but sums to infinity.

Geometric series. In a geometric series, each term is some constant times the preceding term. If we denote the first term a (which we'll assume is not 0) and the ratio of a term to the preceding term by r, then a geometric series has the form

$$a + ar + ar^2 + ar^3 + \cdots .$$

If the ratio r is greater than 1, then the terms approach infinity, so their sum also approaches infinity. If the ratio r is less than -1, then half the terms are positive and approach $+\infty$ and half are negative and approach $-\infty$. In that case, the sum will not approach any number but be alternately positive or negative.

But if the ratio r is small, $|r| < 1$, then the geometric series will converge. We'll find its sum now. Write down the n^{th} partial sum S_n, multiply it by r, and subtract.

$$\begin{array}{rcccccccccccc}
S_n &=& a &+& ar &+& ar^2 &+& \cdots &+& ar^{n-1} && \\
rS_n &=& && ar &+& ar^2 &+& \cdots &+& ar^{n-1} &+& ar^n \\
S_n - rS_n &=& a && && && && &-& ar^n
\end{array}$$

Therefore, $S_n(1 - r) = a(1 - r^{n+1})$. Assuming $r \neq 1$, we find that the n^{th} partial sum is

$$S_n = a\,\frac{1 - r^{n+1}}{1 - r}.$$

Suppose now that $|r| < 1$. We'll show that $\lim_{n\to\infty} S_n = \dfrac{a}{1-r}$ using a few properties of limits that follow from the definition. Note that a and $\dfrac{1}{1-r}$ don't depend on n.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\,\frac{1-r^{n+1}}{1-r} = \frac{a}{1-r}\lim_{n\to\infty}(1-r^{n+1}) = \frac{a}{1-r}\left(1-\lim_{n\to\infty} r^{n+1}\right)$$

Since $|r| < 1$, the powers of r approach 0, that is $\lim_{n\to\infty} r^n = 0$. The argument is similar to that for the geometric series we had above with $r = \frac{1}{2}$. Thus, we've shown the following theorem.

Theorem 10 (Geometric series)**.** The geometric series $a + ar + ar^2 + ar^3 + \cdots$ sums to $\dfrac{a}{1-r}$ when $|r| < 1$. It diverges for other values of r.

This theorem gives our first power series representation of a function $f(x)$. Set $a = 1$ and replace r by x. Then the last theorem says that the function $f(x) = \dfrac{1}{1-x}$ has the power series representation

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } x \in (-1, 1).$$

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