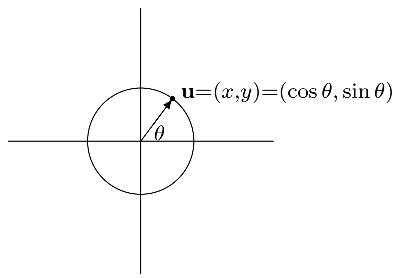
More on Vectors
Math 122 Calculus III
D Joyce, Fall 2012

Unit vectors. A *unit vector* is a vector whose length is 1. If a unit vector $\mathbf{u}$ in the plane $\mathbf{R}^2$ is placed in standard position with its tail at the origin, then it's head will land on the unit circle $x^2+y^2=1$. Every point on the unit circle (x,y) is of the form $(\cos\theta, \sin\theta)$ where θ is the angle measured from the positive x-axis in the counterclockwise direction.



Thus, every unit vector in the plane is of the form $\mathbf{u}=(\cos\theta,\sin\theta)$. We can interpret unit vectors as being directions, and we can use them in place of angles since they carry the same information as an angle.

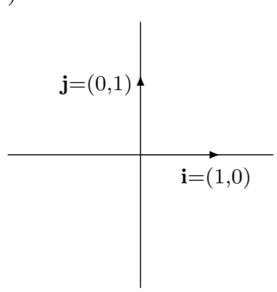
In three dimensions, we also use unit vectors and they will still signify directions. Unit vectors in $\mathbf{R}^3$ correspond to points on the sphere because if $\mathbf{u}=(u_1,u_2,u_3)$ is a unit vector, then $u_1^2+u_2^2+u_3^2=1$. Each unit vector in $\mathbf{R}^3$ carries more information than just one angle since, if you want to name a point on a sphere, you need to give two angles, longitude and latitude.

Now that we have unit vectors, we can treat every vector $\mathbf{v}$ as a length and a direction. The length of $\mathbf{v}$ is $\|\mathbf{v}\|$, of course. And its direction is the unit vector $\mathbf{u}$ in the same direction which can be found by

$$\mathbf{u}=\frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

The vector $\mathbf{v}$ can be reconstituted from its length and direction by multiplying $\mathbf{v}=\|\mathbf{v}\|\,\mathbf{u}$.

The standard bases for $\mathbf{R}^2$ and $\mathbf{R}^3$. Our coordinatized plane $\mathbf{R}^2$ has two standard directions, the x-direction and the y-direction, and we can encode them as unit vectors. We'll denote the unit vector in the x-direction as $\mathbf{i}$ so that $\mathbf{i}=(1,0)$, and the unit vector in the y-direction, as $\mathbf{j}$ so that $\mathbf{j}=(0,1)$.



Every vector $\mathbf{v} = (x, y)$ can be uniquely written as a linear combination of these two standard unit vectors.

$$\mathbf{v} = (x, y) = x\mathbf{i} + y\mathbf{i}$$

We say that $\mathbf{i}$ and $\mathbf{j}$ form the *standard basis* for $\mathbf{R}^2$.

Likewise, for space $\mathbf{R}^3$, there are three standard unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 1, 0)$ in the standard basis. Each vector $\mathbf{v} = (x, y, z) \in \mathbf{R}^3$ is a unique linear combination $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Vectors in dimension n and n-space $\mathbf{R}^n$. So far, we've studied vectors in $\mathbf{R}^2$, primarily because we can draw them easily. But everything we've said about dimension 2 also holds in an arbitrary dimension n.

A vector $\mathbf{v}$ in n-space $\mathbf{R}^n$ can be interpreted as an arrow in $\mathbf{R}^n$ with a certain length and a certain direction. As in the case when $n = 2$, it can be interpreted as lots of different arrows with that length and direction. When it's put in "standard position," the head of the arrow is at a point $(v_1, v_2, \ldots, v_n)$ and the tail of the arrow is at the origin $(0, 0, \ldots, 0)$. Using this standard position, we can identify the vector $\mathbf{v}$ with a point $\mathbf{v} = (v_1, v_2, \ldots, v_n)$.

The standard basis for $\mathbf{R}^n$. One common notation for the standard basis for n-space $\mathbf{R}^n$ is $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ where

$$\begin{array}{rcl} \mathbf{e}_1 & = & (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 & = & (0, 1, 0, \ldots, 0) \\ & \vdots & \\ \mathbf{e}_n & = & (0, 0, 0, \ldots, n) \end{array}$$

The k^{th} standard basis vector $\mathbf{e}_k$ in this basis has 0's in every coordinate except in the k^{th} coordinate there's a 1.

Each vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a unique linear combination of these standard basis vectors

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n = \sum_{k=1}^{n} v_k\mathbf{e}_k.$$

Vector operations. The primary two vector operations are vector addition and multiplication by scalars. Vector addition, $+ : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, is an operation that takes two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in n-space and produces another vector $\mathbf{v} + \mathbf{w}$ in n-space. It's defined coordinatewise by

$$\mathbf{v} + \mathbf{w} = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n).$$

Scalar multiplication, $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, takes a scalar c, that is, a real number, and a vector $\mathbf{v}$ in n-space and produces another vector $c\mathbf{v}$ in n-space. It's also defined coordinatewise.

$$c\mathbf{v} = c(v_1, v_2, \ldots, v_n) = (cv_1, cv_2, \ldots, cv_n)$$

These two operations enjoy the same properties for n-space as they do for 2-space.

Other operations can be defined from these two, namely, negation $-\mathbf{v} = (-1)\mathbf{v}$, and subtraction $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$. They're also computed coordinatewise and have the usual properties.

Norm and dot products are defined for $\mathbf{R}^n$ as well, and have the usual properties.

$$\begin{aligned} \|\mathbf{v}\| &= \|(v_1, v_2, \ldots, v_n)\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\sum_{k=1}^{n} v_k^2} \\ \mathbf{v} \cdot \mathbf{w} &= (v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k \end{aligned}$$

Coordinates for physical space. Frequently, people interpret physical space as $\mathbf{R}^3$. Of course, we know now that the geometry of physical space is not the same as $\mathbf{R}^3$, but close enough for many practical purposes. In order to place a coordinate system on physical space, several choices are required. Different choices lead to different coordinate systems.

(a). Choose a location in physical space to call the origin, $(0, 0, 0)$.

(b). Choose a line through the origin to be the x-axis.

(c). Choose a point on the x-axis to be $(1, 0, 0)$. This choice determines the scale of the coordinate system. The distance between $(0, 0, 0)$ and $(1, 0, 0)$ will be the unit distance.

(d). Choose a line perpendicular to the x-axis to be the y-axis. There are infinitely many to choose from, but they all lie in a plane perpendicular to the x-axis passing through the origin.

(e). There are two points on the y-axis at unit distance from the origin. Choose one of them to be the point $(0, 1, 0)$.

(f). There is one line perpendicular to both the x-axis and the y-axis. Call it the z-axis.

(g). There are two points on this z-axis at unit distance from the origin. Choose one of them to be the point $(0, 0, 1)$. Depending on which one you choose, the resulting coordinate system is called a right-handed coordinate system or a left-handed coordinate system for physical space.

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