# Math 126, Number Theory

Second Test, alternate

11 Apr 2006

Your name: ____________________________

You may use one sheet of prepared notes and a calculator for the test. Points for each problem are in square brackets. Before writing out any proof, please work on scratch paper until you know how the proof goes, then write the proof in the space provided.

**Problem 1.** On Pythagorean triples. [18] Recall that a Pythagorean triple $(x, y, z)$ consists of three positive integers such that $x^2 + y^2 = z^2$. Show that for any Pythagorean triple at least one of $x$, $y$, or $z$ is divisible by 5. [Hint: what are the squares mod 5?]

**Problem 2.** Yes/no. [16; 4 points each part] For each of the following just write "yes" or "no". No explanation is needed unless it's not clear which is correct.

________________ **a.** Note that if $(a, 15) = 1$, then $a^4 \equiv 1 \pmod{15}$. Also note that $\phi(15) = 8$. Does 15 have any primitive roots?

________________ **b.** Fermat's last theorem says that the Diophantine equations $x^n + y^n = z^n$ have no positive solutions for $n > 2$. Did Fermat prove this theorem for any value of $n > 2$ at all?

________________ **c.** If $xy = z^2$ and $x$ and $y$ are relatively prime, then does it follow that each of $x$ and $y$ are perfect squares?

________________ **d.** If $a^4 \equiv 1 \pmod{n}$, then is the order of $a$ modulo $n$ equal to 4?

**Problem 3.** [18] Find at least one positive solution of quadratic Diophantine equation

$$x^2 + xy - 6y^2 = 21.$$

[Hint: factor the left side of the equation.]

**Problem 4.** [15; 5 points each part] On order and primitive roots.

**a.** What is the order of 2 modulo 17?

**b.** Is 2 a primitive root modulo 17?

**c.** How many primitive roots modulo 17 are there?

**Problem 5.** [15] On Euler's $\phi$ function.

**a.** [5] How many positive integers less than 56 are relatively prime to 56?

**b.** [10] Show that if $n > 2$ then $2|\phi(n)$.

**Problem 6.** [18] Solve the pair of linear congruences

$$\begin{cases} 4x + 2y \equiv 3 \pmod{11} \\ 2x - 3y \equiv 8 \pmod{11} \end{cases}$$

Show your work.

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|---|---|
| #1.[18] | |
| #2.[16] | |
| #3.[18] | |
| #4.[15] | |
| #5.[15] | |
| #6.[18] | |
| Total | |