Math 126 Number Theory

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Due Today. From page 32, exercises 5, 7, 12, 13, 15.

Due Wednesday. From page 35, exercises 1, 2.

Due Friday. From page 43, exercises 2, 3, 5, 6, 7.

For next time. Read the rest of chapter 2.

Last meeting. Some applications of the unique factorization theorem including irrationality of surds.

Today. Divisors of a number, their number and their sum. Multiplicative functions.

We're going to look a little more carefully at the divisors of a number n. We've already seen how the divisors fit into a lattice, and we'll be using that lattice today.

Let $d(n)$ denote the number of divisors of n. For example, $n = 24$ has 8 divisors, namely 1, 2, 4, 8, 3, 6, 12, and 24. Therefore $d(24) = 8$. Is there an easier way to determine $d(n)$ than listing all the divisors and counting them? Yes, we'll develop the answer in class.

Let $\sigma(n)$ denote the sum of the divisors of n. For example, the sum of the divisors of $n = 24$ is $1+2+4+8+3+6+12+24 = 60$. Again, we'll develop a way to find $\sigma(n)$ without listing all the divisors of n.

Definition. A function f defined on the natural numbers $\mathbf{N}$ is said to be *multiplicative* if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime.

In a later chapter, we'll look at another multiplicative function, Euler's totient function ϕ. The number of integers between 1 and n that are relatively prime to n is denoted $\phi(n)$. When $n = 24$, those integers relatively prime to n are 1, 5, 7, 11, 13, 17, 19, and 23, so $\phi(24) = 8$.

Perfect numbers. A number is said to be *perfect* if it equals the sum of its proper divisors. In other words, n is perfect if $n = \sigma(n) - n$, that is, $\sigma(n) = 2n$.

Two examples of perfect numbers are 6 and 28. The proper divisors of 6 are 1, 2, and 3, while the proper divisors of 28 are 1, 2, 4, 7, and 14. We'll look at Euclid's Proposition IX.26.

It's amazing how many false statements about perfect numbers were believed for centuries. For instance, it was commonly accepted that perfect numbers alternately ended in 6 and 8, and there was exactly one perfect number of any given number of digits. Frequently, numbers were claimed to be perfect when they weren't.