Math 126 Number Theory

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Due Today. Page 148: 1, 4, and page 151: 3, 11.

Second test. Wednesday, March 8.

Last two meetings. We discussed higher degree Diophantine equations. The statement of the Fermat/Wiles theorem, homogeneous equations, the methods of factoring and congruences to solve Diophantine equations, Pell equations including the following theorems.

> *Theorem.* When a prime p is congruent to 3 modulo 4, then the congruence $x^2 \equiv -1 \pmod{p}$ has no solutions.
>
> *Theorem.* The Pell equation $x^2 - dy^2 = -1$ has no solutions either when 4 divides d or when a prime p divides d where p is congruent to 3 modulo 4.

Next time. Fermat's method of descent. We'll follow Fermat's use of it to prove $x^4 + y^4 = z^4$ has no nontrivial solutions. We may also have time to apply it to Pell equations.

Today. A detailed analysis of Pythagorean triples. Finding all the Pythagorean triples is fun to do, so rather than follow the proof in the text, or Euclid's proof in X.29, we'll try to find one ourselves. That may take longer, but it's worth it.

First, the definition. A *Pythagorean triple* is a triple (x, y, z) of three positive integers

$$x^2 + y^2 = z^2.$$

We'll call this Diophantine equation the *Pythagorean equation.*

A couple of such triples are $(3, 4, 5)$, the most famous one, and $(5, 12, 13)$.

It's clear that you can scale up a Pythagorean triple to get another one. For instance, we can scale $(3, 4, 5)$ by a factor of 10 to get the Pythagorean triple $(30, 40, 50)$. That works because the Pythagorean equation is a homogeneous equation. Conversely, if x, y, and z have a common divisor d, then we can scale down by a factor of d. We define a *primitive Pythagorean triple* to be one where the greatest common divisor (x, y, z) is 1. (Note that we're using the same notation (x, y, z) for a triple and for a GCD. That shouldn't cause any problem because we can tell by the context which is meant in any instance.)

So, our job is to find all of the primitive Pythagorean triples. We'll explore ideas you have to approach the problem. Don't forget to consider techniques we've seen in the course.