Math 126 Number Theory

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A little Pythagorean music. Pythagoras studied musical pitch. If you take a single chord, a *monochord*, and pluck it, one musical note, or pitch, sounds. We'll call that the fundamental. If you hold the chord down against something fixed, then pluck part part of the chord, then a different pitch sounds, a higher one. The shorter the vibrating part of the chord, the higher the pitch.

Pythagoras noted that exactly half of the chord is vibrating, then the sound produced higher, but very similar to the fundamental pitch. Call the note that's produced the second harmonic, what we also call an octave above the fundamental. And if $\frac{1}{}$ of the chord is vibrating, the sound produced is still higher, but very similar to the fundamental and the second harmonic. This fourth harmonic is one octave above the second harmonic. Likewise, the eighth harmonic is an octave above that.

He noted that we can hear the relative pitch of a higher note above a lower note, and if another pair of notes have same relative pitch of the higher to the lower, then the ratios of the lengths of vibrating chords are equal. We can write this mathematically as

$$\frac{w}{x} = \frac{y}{z}$$

where w/x is the ratio of lengths of the first pair of vibrating chords while y/z is the ratio of lengths of the second pair.

Pythagoras didn't stop with the ratio of 2/1, where the second note sounds an octave above the first, but considered other ratios. Most importantly, 3/2. Here, the second note is above the first, but not so far above it as an octave. Still, it's an easily identified interval. It corresponds to what we call a "fifth." An example is the interval from C to the G above it, and other examples are G to D, and F to C. A natural pair of notes in this ratio that comes up on a monochord are the second harmonic ($\frac{1}{2}$ length of the chord) and the third harmonic ($\frac{1}{3}$ of the chord).

An another recognizable interval corresponds to the ratio 4/3. Of course, that's the ratio of the fourth harmonic to the third harmonic. In modern terms, it the interval called a "fourth." Examples are C to F, D to G, and G to C.

The ancient Greeks developed several different sets of notes based on these ratios. When those notes are put in order of pitch, a scale results. We currently have two different kinds of scales, a major scale and a minor scale. When all the notes are put together, the notes can be placed fairly evenly apart and result in a 12-note scale.

A scale built on fifths. Besides the octave with ratio 2/1, let's try to build a scale based on fifths with the ratio of 3/2. We'll give the modern names for the notes and the ratios that result. Let's suppose that C has a length of 1, so that the other notes will have lengths that are reciprocals of the their ratios to C. We'll adjust the lengths by factors of two to keep all the within one octave above C. That means their lengths will be between $\frac{1}{2}$ and 1.

note	length
C	1
G	2/3
D	8/9
A	16/27
E	64/81
B	128/243

We'll stop there, but this could go on forever since the powers $(2/3)^n$ of $\frac{2}{3}$ are incommensurable with the powers of $\frac{1}{2}$. We certainly don't want an infinite scale, so we'll have to do some approximating. After all, our ears aren't perfect.

The question is, when is some power of $\frac{2}{3}$ about the same as some power of $\frac{1}{2}$?

$$(2/3)^n \approx (1/2)^m$$

Take logs base 2 of both sides of the equation and simplify.

$$\begin{aligned} n \log_2(2/3) &\approx m \log_3(1/2) \\ n(1 - \log_2 3) &\approx -m \\ \log_2 3 &\approx m/n \end{aligned}$$

Thus, we want to approximate the number $\log_2 3$ by a rational number.

Let's find the continued fraction expansion of $\log_2 3 = 1.5849625$.

$$\log_2 3 = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\cdots$$

The first few rational approximates of this number are

$$1, 2, \frac{3}{2}, \frac{7}{4}, \frac{19}{12}, \frac{65}{41}.$$

So, good candidates for n, the number of notes in a scale based on fifths, are 2, 4, 12, or 41. It is the 12 here that our 12-tone scale is based on. Note that $\frac{19}{12} = 1.58\overline{3}$ is pretty close to $\log_2 3$. Of course $\frac{65}{41} = 1.58536...$ is closer.

So, why not 41 notes to a scale instead of 12?

Other considerations. The major triad—for example C, E, G—is very strong in our music. It could be argued that what we're hearing is a ratio based on lengths $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{6}$, that is, the note E should sound like the fifth harmonic. Pythagoras, apparently, didn't use the fifth harmonic, but that doesn't mean our music doesn't use it. If it does, then we should include $\log_2 5$ in our analysis.