Math 126 Number Theory

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Solutions to Pell equations by continued fractions. Recall that a Pell equation is of the form

$$x^2 - dy^2 = \pm 1.$$

We'll assume that d is not a square.

The solutions to this equation are very closely related to the continued fraction expansion of $\sqrt{d}$.

If p/q is a rational approximation of $\sqrt{d}$, then p^2/q^2 is close to d, so $p^2/q^2 - d$ is near zero, and so $p^2 - dq^2$ is small. Thus, $x = p$ and $y = q$ is a candidate for a solution to the Pell equation $x^2 - dy^2 = \pm 1$.

Let's take an example. The continued fraction expansion of $\sqrt{51}$ is

$$\sqrt{51} = 7 + \frac{1}{7+}\,\frac{1}{14+}\,\frac{1}{7+}\,\frac{1}{14+}\cdots.$$

There's an easy way to compute the rational approximates given the continued fraction that's illustrated in the following table. The first two rows are dummy rows to get started. They're always the same. The next row starts with $a_0 = 7$, the integer part of $\sqrt{51}$. The values of q_n and p_n are recursively defined by the equations

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2} \\ p_n &= a_n p_{n-1} + p_{n-2} \end{aligned}$$

so that, for $\sqrt{51}$, we have, for instance, $q_1 = a_1 q_0 + q_{-1} = 7\cdot 1 + 0 = 7$, and $p_1 = a_1 p_0 + p_{-1} = 7\cdot 7 + 1 = 50$.

n	a_n	q_n	p_n	p_n/q_n
-2		1	0	
-1		0	1	
0	7	1	7	$7 = 7.0000000000$
1	7	7	50	$50/7 = 7.1428571428$
2	14	99	707	$707/99 = 7.1414141414$
3	7	700	4999	$4999/700 = 7.1412857128$
4	14	9899	70693	7.1414284271
5	7	69993	499850	7.1414284285

Note how these approximates are getting closer to $\sqrt{51} = 7.14142842854\ldots$.

But look at the values of $p_n^2 - 51q_n^2$.

n	p_n	q_n	$p_n^2 - 51q_n^2$
1	7	1	-2
2	50	7	1
3	707	99	-2
4	4999	700	1
5	70693	9899	-2

We've found some solutions to the Pell equation $x^2 - 51y^2 = 1$:) But we didn't find any to the other Pell equation $x^2 - 51y^2 = -1$:(

So, what's the theorem? When can we solve the Pell equations $x^2 - dy^2 = \pm 1$? You can always solve $x^2 - dy^2 = 1$, and there are infinitely many solutions, and they're all every other rational approximate given by the continued fraction expansion of $\sqrt{d}$. But the other Pell equation $x^2 - dy^2 = -1$ only has solutions when the continued fraction expansion of $\sqrt{d}$ has an odd period, and in that case, they're the remaining rational approximates of the continued fraction expansion of $\sqrt{d}$.