Math 128 Geometry
Final Exam

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Write your answers on lined paper. Staple the pages together. Show all your work for credit. You may use notes and textbooks for this exam, but don't talk to anyone but me about the test (until it's over). Point values for each problem are in square brackets.

Be sure that your proofs and computations are easy to read.

Problem 1. [16] **On transformation groups.** Let L be a line in the Euclidean plane. Consider the group G of isometries T of the plane that leave L invariant. Such an isometry preserves distance, and for each point P on L, $T(P)$ also lies on L.

a. As you recall, the isometries of the plane come in four types: translations, rotations, reflections, and glide reflections. Are each of these four types represented in G? For each of the four types, if there are any transformations of that type in G, give an example. (Please exclude the identity transformation from consideration even though it is both a translation and rotation.)

b. Show that this group G is not commutative. Select two transformations in G that don't commute and explain why they don't.

Problem 2. [20] **Some philosophical implications.**

For the longest time Euclidean geometry was considered to be the geometry of actual physical space. Kant (1724-1804), a very influential philosopher, stated (1781) "the concept of [Euclidean] space is by no means of empirical origin, but is the inevitable necessity of thought." Fifty years later, Bolyai, Lobachevsky, and Gauss had accepted hyperbolic geometry as being as fully consistent as Euclidean geometry. Yet as late as 1888 Lewis Carroll was poking fun at non-Euclidean geometry.

Write an essay (a couple pages) about the geometry of physical space. Explain in your own words what the relation is between mathematical geometry and physical space. Here are a few questions you might consider: Is physical space Euclidean? Is it hyperbolic? Elliptic? Something else? How can you tell? When is a curve in physical space a straight line? What is distance?

There should be no fluff in your essay. Organize your essay well, and make your points as clearly as you can.

Problem 3. [18; 6 points each part] **An isometry.** Consider the isometry T of the Euclidean plane $\mathbf{C}$ which sends a point $x + iy$ in the plane to the point $(1 - y) + i(1 + x)$.

a. Determine whether T is a translation, a reflection, a rotation, or a glide reflection. Justify your claim.

b. Does T have any fixed points? If so, what are they?

c. Does T preserve or reverse orientation? Why?

Problem 4. [20; 2 points each part] **Comparison of geometries.**

For each of the following five statements, and for each of the following two geometries, explain why the statement is or is not a theorem of the geometry. A single sentence will do in each case. Your answers will probably fall into three broad categories: the statement is true in the geometry, the statement is false in the geometry, or the statement can't be expressed in the geometry. No proofs are necessary for this problem.

(a). The sum of the three angles of a triangle is equal to two right angles.

(b). In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(c). If line l does not intersect line m, and line m does not intersect line n, then line l does not intersect line n.

(d). If the three angles of triangle ABC equal respectively the three angles of triangle $A'B'C'$, then triangle ABC is congruent to triangle $A'B'C'$.

(e). For any two lines l and m, there is a unique line n which is perpendicular to both l and m.

(i). Plane Euclidean geometry

(ii). Plane hyperbolic geometry

Problem 5. Angle defect. [16] In hyperbolic geometry, the angle sum of a triangle is less than 180 degrees, the amount less than 180 degrees being called the angle deficit.

Let ABC be an equilateral triangle, that is, a triangle where the three sides are all equal. Let D be the center of triangle ABC. Note that this point D can be defined in various ways, for instance, the intersection of the angle bisectors of the triangle, the intersection of the altitudes of the triangle, the intersection of the medians of the triangle, or the intersection of the perpendicular bisectors of the sides of the triangle.

Explain why the angle defect of the triangle ABD is one-third of the angle defect of the triangle ABC.

Problem 6. [10; 5 points each part] On congruent figures in a geometry. [20] Recall that two figures are congruent with respect to a transformation group if there is a transformation in that group that maps one of the figures to the other one. Consider the Möbius transformation group G for Möbius geometry.

Let A be the figure consisting of three clines: the x-axis, the y-axis, and the circle of radius 1 centered at $z = 1$. Note that all three pass through the origin 0, and two of them pass through the point at ∞.

a. Draw or describe a figure B congruent to A under G which does not include the point at ∞.

b. Draw or describe a figure C congruent to A under G where the point 0 in A corresponds to the point ∞ in B.