Math 128, Modern Geometry

D. Joyce, Clark University

17 Oct 2005

Due Wednesday. From Chapter 7: 1, 6, 9, 10.

Read for next time. Chapter 9 on hyperbolic length. We won't cover chapters 9 and 10 in detail since our course does not have calculus as a prerequisite. We'll treat them as summaries of the topics and skip the proofs.

Start chapter 9 with the distance formulas. The first says that the distance from 0 to a real number r between 0 and 1 is $d(0,r) = \ln\left(\dfrac{1+r}{1-r}\right)$ where ln stands for the natural logarithm, that is to say, the logarithm to the base e. The second is more general and gives the distance between two points z_1 and z_2.

$$d(z_1, z_2) = \ln\left(\frac{1+r}{1-r}\right) \text{ where } r = \left|\frac{z_2 - z_1}{1 - z_1\overline{z}_2}\right|$$

The important thing about this distance is that all the transformations of the hyperbolic plane preserve it. After the section about the distance formulas, read the one about fundamental properties of distance, and skim the rest of the chapter.

Last time. Hyperbolic parallels, circles, horocycles, and hypercycles.

Today. Geometric classification of transformations of the hyperbolic plane. We'll base this classification on the one we did for Möbius transformations. The circles, horocycles, and hypercycles we talked about last time come into it.

Recall the classification of Möbius transformations. First of all, a Möbius transformation has exactly two fixed points, or it has exactly one fixed point, or it's the identity transformation which fixes all points.

Those with two fixed points come in three kinds. (1) There are the hyperbolic transformations which move points along the circles in the family of Steiner circles of the first kind that pass through the two fixed points. (2) There are the elliptic transformations which move points along the circles in a family of Steiner circles of the second kind. (Those are the circles $\perp$ to the just-mentioned family of Steiner circles of the first kind.) (3) There are the loxodromic transformations that don't move points along any circles, but rather along spirals going from one fixed point to the other fixed point.

Besides the transformations with two fixed points, there are transformations with exactly one fixed point. (4) They are the parabolic transformations. They move points along circles in a family of degenerate Steiner circles. They're all the circles tangent to one straight line passing through the fixed point.

Hyperbolic transformations. Now assume that the Möbius transformation is a transformation of the hyperbolic plane, $\mathbf{D}$. That means it sends the unit circle $\partial\mathbf{D}$ to itself.

(1) Can it be a hyperbolic transformation that moves points along the circles in the family of Steiner circles of the first kind that pass through two fixed points p and q? Yes, but only if the unit circle $\partial\mathbf{D}$ is one of those Steiner circles. (Otherwise it wouldn't send the unit circle to itself.) That means the two fixed points p and q are two ideal points on $\partial\mathbf{D}$. The rest of the Steiner circles intersect $\mathbf{D}$ in hypercycles that pass through p and q, except the one that is $\perp \partial\mathbf{D}$, and that's the hyperbolic straight line connecting p to q. This transformation is called a *hyperbolic translation* along that

line. Unlike Euclidean translations, a hyperbolic translation does *not* translate along parallel lines, too; it just translates along that one line.

(2) Can it be an elliptic transformation that moves points along the circles in a family of Steiner circles of the second kind, the circles $\perp$ to all the circles passing through the two fixed points p and q? Yes, but only if the unit circle $\partial\mathbf{D}$ is one of those Steiner circles of the second kind. Then one of the two points, say p lies in $\mathbf{D}$, and the other outside. In fact, the other will be the dual point p^*, the inversion of p in the unit circle. (This dual point is not the reciprocal $1/p$, but what our text calls the symmetric point $1/\overline{p}$.) The rest of the Steiner circles of the second kind in $\mathbf{D}$ are hyperbolic circles between p and $\partial\mathbf{D}$. The point p is the hyperbolic center of these circles, and the transformation is called a *hyperbolic rotation* around p.

(3) Can it be a loxodromic transformation? No. Loxodromic transformations don't send any circles to themselves, so they can't send $\partial\mathbf{D}$ to itself.

(4) Can it be a parabolic transformation? Yes, if $\partial\mathbf{D}$ is one of the circles in the family of degenerate Steiner circles that the parabolic transformations moves points along. In that case the single fixed point p is an ideal point on $\partial\mathbf{D}$ and the rest of the circles in the family that are inside $\mathbf{D}$ are are all tangent to $\partial\mathbf{D}$ at p. In other words, they're all horocycles. These transformations are called *parallel displacements.*

Summary. Besides the identity transformation, the transformations of the hyperbolic plane are

hyperbolic translations along one line in the plane that move points along hypercycles,

hyperbolic rotations around a point that move points along circles, or

parallel displacements that move points along horocycles.