

**Math 130 Linear Algebra**  
**Final Exam Answers, Dec 2006**

Scale: 80–100 A, 65–79 B, 50–64 C. Median 63.

**Problem 1.** [14; 7 points each part] Let  $A$  be the matrix

$$A = \begin{bmatrix} 2 & 5 & 0 & 2 & -2 \\ 0 & 2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 3 & 14 \end{bmatrix}.$$

**a.** Recall that the row space of an  $m \times n$  matrix is the subspace of  $\mathbf{R}^n$  spanned by the rows of the matrix. Find a basis for the row space for  $A$ .

The matrix row reduces to  $\begin{bmatrix} 1 & 2.5 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1.5 & 2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ , so

the row space has dimension 4. You can take either the original 4 rows, these 4 rows, or the rows you get when you put the matrix in reduced echelon form, that is,  $\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$ , as a basis.

**b.** Recall that the null space of a matrix  $A$  is the set of all solutions of the homogeneous system  $A\mathbf{x} = \mathbf{0}$ . Find a basis for the null space for  $A$ .

Since the rank of the matrix is 4, therefore the nullity is 1, so there needs to be one vector, and it's clear that it's  $(0, 0, 1, 0, 0)$ .

**Problem 2.** [14; 7 points each part] Consider the three vectors  $\mathbf{u} = (1, 3, 1)$ ,  $\mathbf{v} = (4, 2, -1)$ , and  $\mathbf{w} = (-3, 1, 2)$ .

**a.** Either prove that the  $\mathbf{u}$  is in the span of the vectors  $\mathbf{v}$  and  $\mathbf{w}$ , or prove that it is not.

By inspection you can see that  $\mathbf{u} = \mathbf{v} + \mathbf{w}$ , so  $\mathbf{u}$  is a linear combination of  $\mathbf{v}$  and  $\mathbf{w}$ , that is, it's in their span.

**b.** Are the three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  linearly dependent, or linearly independent?

If one of them is a linear combination of the others, then they're linearly dependent.

**Problem 3.** [10] A parallelogram in  $\mathbf{R}^3$  has as adjacent sides the vectors  $\mathbf{u} = (1, 3, -2)$  and  $\mathbf{v} = (3, -1, -1)$ . Determine the area of the parallelogram.

The area of the parallelogram is the length of the cross product of  $\mathbf{u}$  and  $\mathbf{v}$ . Since  $\mathbf{u} \times \mathbf{v} = (-5, -5, -10)$ , therefore that area is  $\sqrt{25 + 25 + 100} = \sqrt{150}$ .

**Problem 4.** [20; 5 points each part] Let  $A$  be the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

**a.** Write down the characteristic polynomial  $f(\lambda)$  for  $A$ .

$$f(\lambda) = \begin{vmatrix} -\lambda & -1 & 2 \\ 1 & -\lambda & 1 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = (\lambda^2 + 1)(3 - \lambda)$$

**b.** Determine the eigenvalues for  $A$ .

$$\lambda = i, -i, 3.$$

**c.** For each of the eigenvalues of  $A$ , find the eigenspace of eigenvectors for that eigenvalue.

First,  $\lambda = i$ . Solve the matrix equation  $(A - iI)\mathbf{x} = \mathbf{0}$ , that is,

$$\begin{bmatrix} -i & -1 & 2 \\ 1 & -i & 1 \\ 0 & 0 & 3 - i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

From the third equation,  $(3 - i)z = 0$ , we see  $z = 0$ . The first two equations reduce to the same thing, namely,  $x = iy$ . Therefore, the eigenvectors are  $(iy, y, 0)$  for any value of  $y$ .

Second,  $\lambda = -i$ , leads to eigenvectors  $(x, y, z) = (-iy, y, 0)$ .

Third,  $\lambda = 3$ . Solve the matrix equation  $(A - 3I)\mathbf{x} = \mathbf{0}$ , that is,

$$\begin{bmatrix} -3 & -1 & 2 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

This implies  $x = \frac{1}{2}z$  and  $y = \frac{1}{2}z$ , so the eigenvectors are  $(x, y, z) = (\frac{1}{2}z, \frac{1}{2}z, z)$  for arbitrary  $z$ .

**d.** Is  $A$  a diagonalizable matrix? Explain why or why not.

Yes it is, at least over  $\mathbf{C}$ , since the eigenvalues are distinct. If it's required that the matrix entries are real, however, then it's not diagonalizable.

**Problem 5.** [20; 10 points each part] Recall that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if (1)  $\mathbf{0}$  is a vector in  $W$ , (2)  $W$  is closed under vector addition, and (3)  $W$  is closed under scalar multiplication.

**a.** Prove that the intersection

$$W_1 \cap W_2 = \{\mathbf{v} \in V \mid \mathbf{v} \in W_1 \text{ and } \mathbf{v} \in W_2\}$$

of any two subspaces  $W_1$  and  $W_2$  of a vector space  $V$  is also a subspace of  $V$ .

(1). Since  $\mathbf{0}$  is a vector in both  $W_1$  and  $W_2$ , therefore it's in their intersection.

(2). Suppose that  $\mathbf{v}$  and  $\mathbf{w}$  are both vectors in  $W_1 \cap W_2$ . Then they're both vectors in each of  $W_1$  and  $W_2$ . Since  $W_1$  and  $W_2$  are each closed under vector addition, therefore the sum  $\mathbf{v} + \mathbf{w}$  is a vector in both  $W_1$  and  $W_2$ . Therefore, it's a vector in their intersection  $W_1 \cap W_2$ . Thus,  $W_1 \cap W_2$  is closed under vector addition.

(3). Suppose that  $\mathbf{v}$  is a vector in  $W_1 \cap W_2$  and  $c$  is a scalar. Since  $\mathbf{v}$  is a vector in each of  $W_1$  and  $W_2$ , and they're both closed under scalar multiplication, therefore  $c\mathbf{v}$  is a vector in each of  $W_1$  and  $W_2$ . Therefore, it's a vector in their intersection  $W_1 \cap W_2$ . Thus,  $W_1 \cap W_2$  is closed under scalar multiplication.

**b.** Give an example that shows that the union of two subspaces does not have to be a subspace. For your example,

specify what the vector space  $V$  is, what the two subspaces  $W_1$  and  $W_2$  of  $V$  are, and explain why the union

$$W_1 \cup W_2 = \{\mathbf{v} \in V \mid \mathbf{v} \in W_1 \text{ or } \mathbf{v} \in W_2\}$$

is not another subspace of  $V$ .

Probably the simplest example is to take  $V$  to be  $\mathbf{R}^2$ ,  $W_1$  to be the  $x$ -axis, and  $W_2$  to be the  $y$ -axis. Then the union  $W_1 \cup W_2$  consists of the  $x$ -axis and the  $y$ -axis but nothing else. Since  $(1, 0)$  and  $(0, 1)$  belong to the union, but their sum  $(1, 1)$  does not, therefore the union is not closed under vector addition. Thus, the union is not a subspace.

Note that sometimes  $W_1 \cup W_2$  is a subspace of  $V$ , so an example is required to show that the union of subspaces is not always a subspace.

**Problem 6.** [12; 4 points each part] On dimension and basis. Let  $V$  be the vector space  $V = \{(w, x, y, z) \in \mathbf{R}^4 \mid w = x + y + z\}$ .

**a.** What is the dimension of  $V$ ? Explain how you know that dimension.

It's 3. There are various intuitive reasons that suffice, but a rigorous one is to say that the null space  $V$  of the  $1 \times 4$  matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix}$$

has dimension 3 since the matrix has rank 1.

**b.** Exhibit a basis for  $V$ . (No need to explain how you found it.)

You probably found the canonical solution to  $A\mathbf{x} = \mathbf{0}$  which has a basis of

$$(-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1).$$

**c.** Give an example of a 2-dimensional subspace  $W$  of  $V$ .

There are many such subspaces. For instance, the subspace spanned by two of the vectors in part b.

**Problem 7.** [10] If  $A$  is a  $3 \times 5$  matrix, show that the columns of  $A$  are linearly dependent.

The row rank has to be less than or equal to 3, the number of rows. But the row rank equals the column rank, so the column rank has to be less than or equal to 3. Therefore at most 3 of the columns can be linearly independent. There being 5 columns, they must be linearly dependent.