

Math 130 Linear Algebra First Test Answers, Sep 2006

Scale: 85–100 A, 73–84 B, 60–72 C. Median 73.

Problem 1. [15] The following matrix describes a system of linear equations in five unknowns— v, w, x, y, z . What is the general solution to this system?

$$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \end{array} \right]$$

Three of the variables, v , w , and y , are determined by the values of the remaining two variables, x and z . Precisely, $y = -2 - 7z$, $w = 8 - 4z$, and $v = 4 - 5w - 2y + 2z = -32 + 36z$. You can describe the general solution as

$$(v, w, x, y, z) = (-32 + 36z, 8 - 4z, x, -2 - 7z, z)$$

where x and z are arbitrary.

Problem 2. [10] Recall that a square matrix A is symmetric if it equals its own transpose, that is, $A^T = A$. Prove that for every square matrix B , it is the case that $A = B + B^T$ is a symmetric matrix.

You can explain this either in terms of elements of the matrix or algebraically using matrix algebra.

Here's a proof that displays the elements of the matrices. Let

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

Then

$$B^T = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{n1} \\ b_{21} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

Therefore

$$\begin{aligned} A &= (B + B^T) \\ &= \begin{bmatrix} 2b_{11} & b_{12} + b_{21} & \cdots & b_{1n} + b_{n1} \\ b_{21} + b_{12} & 2b_{22} & \cdots & b_{2n} + b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} + b_{1n} & b_{n2} + b_{2n} & \cdots & 2b_{nn} \end{bmatrix}. \end{aligned}$$

And, since $b_{ij} + b_{ji} = b_{ji} + b_{ij}$, A is a symmetric matrix.

Here's a proof using matrix algebra. Since $A = B + B^T$, therefore

$$A^T = (B + B^T)^T = B^T + (B^T)^T = B^T + B = A.$$

Thus, A is a symmetric matrix.

Problem 3. [12; 6 points each part] Let \mathbf{x} be an n -vector.

a. Is it possible for the dot product $\mathbf{x} \cdot \mathbf{x}$ to be negative? Explain.

No, it's not possible. If $\mathbf{x} = (x_1, x_2, \dots, x_n)$, then $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2$. A sum of squares has to be nonnegative since each square has to be nonnegative.

b. If $\mathbf{x} \cdot \mathbf{x} = 0$, what is \mathbf{x} ? Explain.

If $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$, then each x_i has to be 0 because the only way a sum of squares can equal 0 is if each is 0 itself. Thus \mathbf{x} is the $\mathbf{0}$ -vector.

Problem 4. [12] Compute the inverse of the 3×3 matrix A by reducing the 3×6 matrix $[A|I]$ to reduced echelon form. (If it turns out the inverse doesn't exist, explain why you concluded that.)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array} \right]$$

Form the 3×6 matrix $[A|I]$, and row reduce it. Here's one sequence of steps to turn it into reduced echelon form. You probably used a different sequence.

$$\begin{aligned} [A|I] &= \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \end{array} \right] \end{aligned}$$

Since there is a row of 0's in the left side of this matrix, therefore A^{-1} does not exist.

Problem 5. [12] In this problem you will consider permutations of the set $S = \{1, 2, 3, 4, 5, 6, 7\}$.

a. [6] How many inversions does the permutation 7531246 have?

The only times an integer precedes a smaller integer are these:

- 7 precedes 5, 3, 1, 2, 4, and 6;
- 5 precedes 3, 1, 2, and 4; and
- 3 precedes 1 and 2.

Therefore, there are 12 inversions.

b. [3] Give an example of an odd permutation of S .

2134567 is an odd permutation since it has an odd number of inversions. Of course, there are many others. Indeed, there are $7!/2 = 2520$ odd permutations.

c. [2] Give an example of an even permutation of S .

7531246 is an even permutation since it has 12 inversions. Of course, there are many others, the identity 1234567 being one of them.

Problem 6. [12] Give a geometric description of the matrix transformation $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $f(\mathbf{u}) = A\mathbf{u}$ when $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

The point (x, y) is mapped to the point $(x/2, y/2)$ half as far away from the origin. This is a contraction by a factor of $\frac{1}{2}$.

Problem 7. [12] Compute the following determinant. (Use whatever method you like, but show all your work.)

$$\begin{vmatrix} 5 & 3 & 1 \\ 0 & -2 & 1 \\ 4 & 1 & 2 \end{vmatrix}$$

This is only a 3×3 matrix, so the definition isn't too much work to use. The determinant equals

$$5 \cdot (-2) \cdot 2 + 3 \cdot 1 \cdot 4 + 1 \cdot 0 \cdot 1 - 1 \cdot (-2) \cdot 4 - 5 \cdot 1 \cdot 1 - 3 \cdot 0 \cdot 2$$

which is -5 .

Problem 8. [15] Prove that if A and B are invertible square matrices of the same size, then AB is also invertible.

The problem here is to construct a matrix C such that $(AB)C = C(AB) = I$. Then C will be the inverse of AB . We've seen that $B^{-1}A^{-1}$ should be that C , so we have to show $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$.

Suppose that A^{-1} and B^{-1} exist. Then

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= ABB^{-1}A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

Likewise,

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}A^{-1}AB \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I \end{aligned}$$

Therefore, $B^{-1}A^{-1}$ is the inverse of AB . QED