Math 130 Linear Algebra Second Test Answers, Nov 2006

Scale: 87–100 A, 75–86 B, 55-7-4 C. Median 80.

Problem 1. On cofactor expansion to evaluate determinants [10]

Use cofactor expansion to evaluate the following determinant. (Your choice of which row or column to use.) Show your work.

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 5 & 0 & 2 & 3 \end{vmatrix}$$

Any row or column will work. Here are the computations when the first row is chosen.

$$1\begin{vmatrix} 0 & 0 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 5 & 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 5 & 0 & 3 \end{vmatrix} - 0$$

$= 1(-2) - 2(18-15) + 3(-6+5) = -11.$

Problem 2. On orthogonality in $\mathbf{R}^n$ [20]

Prove that if a vector $\mathbf{w}$ is orthogonal to both vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$, then $\mathbf{w}$ is orthogonal to any linear combination $r\mathbf{u} + s\mathbf{v}$ of them.

Since $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{w} \perp \mathbf{v}$, therefore $\mathbf{w}\cdot\mathbf{u} = 0$ and $\mathbf{w}\cdot\mathbf{v} = 0$. Thus,

$$\begin{aligned} \mathbf{w}\cdot(r\mathbf{u}+s\mathbf{v}) &= \mathbf{w}\cdot(r\mathbf{u}) + \mathbf{w}\cdot(s\mathbf{v}) \\ &= r(\mathbf{w}\cdot\mathbf{u}) + s(\mathbf{w}\cdot\mathbf{v}) \\ &= r0 + s0 = 0 \end{aligned}$$

Hence, $\mathbf{w} \perp (r\mathbf{u}+s\mathbf{v})$.

Problem 3. On linear transformations [10]

Represent this linear transformation with a matrix.

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+z \\ y+z \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ x+z \\ y+z \end{bmatrix}$$

Problem 4. On areas and volumes [20; 10 points each part]

a. Compute the area of the parallelogram whose vertices are located at

$$(2,3,1), (4,5,2), (6,3,-1), \text{ and } (8,5,0).$$

Two sides of this parallelogram are the difference vectors $\mathbf{u} = (4,5,2) - (2,3,1) = (2,2,1)$ and $\mathbf{v} = (6,3,-1) - (2,3,1) = (4,0,-2)$, so the area is the length of $\mathbf{u}\times\mathbf{v}$.

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 4 & 0 & -2 \end{vmatrix} \\ &= -4\mathbf{i} - 8\mathbf{j} - 8\mathbf{k}. \end{aligned}$$

Therefore, the area is $\sqrt{4^2+8^2+8^2} = \sqrt{144} = 12$.

b. Compute the volume of the parallelepiped with one vertex at the origin, and edges

$$\mathbf{u} = (3,1,-1), \mathbf{v} = (0,2,6), \text{ and } \mathbf{w} = (2,5,3).$$

The volume is the absolute value of the triple scalar product, where the triple product is the determinant

$$\begin{vmatrix} 3 & 1 & -1 \\ 0 & 2 & 6 \\ 2 & 5 & 3 \end{vmatrix} = -56.$$

Therefore, the volume is 56.

Problem 5. [20] Let $\mathbf{u} = (3,4,5)$ and $\mathbf{v} = (2,-1,0)$. Determine the following.

a. $||\mathbf{u}|| = \sqrt{3^2+4^2+5^2} = \sqrt{50}$, which you can also write as $5\sqrt{2}$.

b. $\mathbf{u}\cdot\mathbf{v} = 3\cdot 2 + 4\cdot(-1) + 5\cdot 0 = 2.$

c. $\mathbf{u}\times\mathbf{v}$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 5 \\ 2 & -1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 4 & 5 \\ -1 & 0 \end{vmatrix}\mathbf{i} + \begin{vmatrix} 3 & 5 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}\mathbf{k} \\ &= 5\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}, \end{aligned}$$

which you can also write as $(5, 10, -11)$.

d. A unit vector in the same direction as $\mathbf{u}$.

$$\frac{\mathbf{u}}{||\mathbf{u}||} = \frac{(3,4,5)}{\sqrt{50}} = \left(\frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}}\right).$$

e. The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||} = \frac{2}{\sqrt{50}\,\sqrt{5}}$$

which you can also write as $\dfrac{2}{5\sqrt{10}}$.

Problem 6. On abstract vector spaces [20]

Recall the axioms for vector spaces summarized here. A *vector space* is a set equipped with operations of vector addition and scalar multiplication such that

(a) Vector addition is commutative,

(b) Vector addition is associative,

(c) $\mathbf{0}$ is the identity for vector addition,

(d) Each vector has a negation,

(e) Scalar multiplication distributes over vector addition: $c(\mathbf{v}+\mathbf{w}) = c\mathbf{v} + c\mathbf{w}$,

(f) Scalar multiplication distributes over real addition: $(c+d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$,

(g) Multiplication and scalar multiplication associate: $c(d\mathbf{v}) = (cd)\mathbf{v}$, and

(h) 1 is the identity for scalar multiplication.

Using only these axioms prove that $c\mathbf{0} = \mathbf{0}$ for every scalar c. (You may not assume that the vector space has coordinates, so don't use coordinates for this proof. Mention every time your proof uses any axiom. Hint: sometime in this proof you'll need distributivity.)

There are many proofs you could come up with. Here's a fairly straightforward proof, but not the shortest one. First of all,

$$\mathbf{0} = \mathbf{0} + \mathbf{0}$$

by (c). Now, multiply each side of that equation on the left by the scalar c to get

$$c\mathbf{0} = c(\mathbf{0} + \mathbf{0}).$$

Rewrite the right hand side of that equation using (e) to get

$$c\mathbf{0} = c\mathbf{0} + c\mathbf{0}.$$

Add the negation of $c\mathbf{0}$, which exists by (d), to each side of the equation to get

$$c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0}).$$

Rewrite the right hand side using (b) to get

$$c\mathbf{0} + (-c\mathbf{0}) = c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0})).$$

Now, since $c\mathbf{0} + (-c\mathbf{0}) = \mathbf{0}$, by (d), therefore the equation simplifies to

$$\mathbf{0} = c\mathbf{0} + \mathbf{0}.$$

Finally, using (c), again, on the right hand side, conclude that

$$\mathbf{0} = c\mathbf{0}.$$

That finishes the proof.

This proof used axioms (b), (c), (d), and (e). Other proofs might use other axioms.