Math 130 First Test Answers. Oct 2009. Grade ranges: 90–100 A, 74–89 B, 60–73 C. Median 88.

Problem 1. [10; 5 points each part] On systems of linear equations and their matrices.

a. Exhibit a system of 3 equations in 5 unknowns whose general solution is

$$(v, w, x, y, z) = (4 - 3x + 2z, x - z, x, z - 1, z)$$

where x and z may be any real numbers.

You can write the system as $v = 4 - 3x + 2z$, $w = x - z$, $y = z - 1$, or, in preparation for part **b** as

$$\left\{\begin{array}{rcrcrcrcr} v & & & + & 3x & & & - & 2z & = & 4 \\ & & w & - & x & & & + & z & = & 0 \\ & & & & & & y & - & z & = & -1 \end{array}\right.$$

b. Write the augmented matrix for the system of equations that you gave in part **a**.

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 3 & 0 & -2 & 4 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

Problem 2. [25] Computations. Show your work for credit.

a. [10] Compute the inverse of the matrix A or show that the inverse doesn't exist.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$$

The easiest way is to row reduce $[A|I]$.

$$\begin{aligned} [A|I] &= \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right] \end{aligned}$$

Since the left half has a row of zeros, A has no inverse.

b. [10] Compute the determinant of the matrix B.

Here's one computation that finds the determininant by using row operations to make it a triangular matrix.

$$\begin{aligned} |B| &= \begin{vmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 8 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 8 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 8 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -5/3 \end{vmatrix} \\ &= 1 \cdot (-4) \cdot 3 \cdot (-5/3) = 20 \end{aligned}$$

c. [5] Compute the dot product $\mathbf{v} \cdot \mathbf{w}$ of $\mathbf{v} = (2, 2, -1)$ and $\mathbf{w} = (5, 3, 2)$.

$$\mathbf{v} \cdot \mathbf{w} = 2 \cdot 5 + 2 \cdot 3 + (-1) \cdot 2 = 14.$$

Problem 3. [12; 4 points each part] In this problem you will consider the permutation 23541 of the set $S = \{1, 2, 3, 4, 5\}$.

a. Write the 5×5 permutation matrix for the permuation 23541.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b. Identify the inversions of the permutation 23541.

Here's a list of the pairs that are out of order: 21, 31, 43, 51, and 41. There are 5 of them.

c. Is 23541 an even permutation or an odd permutation?

It's odd since 5 is odd.

Problem 4. [10] Explain why it is the case that when an $n \times n$ matrix A has only zeros in some row, then A has no inverse. Your explanation should be valid for all values of n, not just $n = 2$ or $n = 3$.

You can explain this in various ways. Here's one. If A has all 0s in one row, then AB also has all 0s in that row, no matter what B is. Therefore, AB cannot be the identity matrix I. Hence, A cannot have an inverse.

Here's a different proof. If A has all 0s in one row, then its determinant is 0 since the determinant is the sum of terms, each term being a product factors one factor being a 0 taken from that row. But since $|A| = 0$, it has no inverse.

Problem 5. [10] On linear transformations of the plane.

a. [6] Give a geometric description of the matrix transformation $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f(\mathbf{u}) = A\mathbf{u}$ when $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$.

The vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is transformed to the vector $\begin{bmatrix} 2y \\ 2x \end{bmatrix}$, or as ordered pairs, $(x, y) \mapsto (2y, 2x)$. Geometrically, in terms of what happens to points, a point is reflected across the line $y = x$ then sent twice as far away from the origin. You could also say this is a compostion of the reflection across $y = x$ and an expansion by a factor of 2.

b. [4] Does this tranformation preserve distances?

No, distances are doubled.

Problem 6. [16] A square matrix $A = [a_{ij}]$ is said to be *skew symmetric* if $A^T = -A$. In other words, for all indices i and j, $a_{ij} = -a_{ji}$. For example,

$$\begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & -7 \\ -5 & 7 & 0 \end{bmatrix}$$

is a 3×3 skew symmetric matrix.

Prove that if A is a square matrix, then $B = A - A^T$ is a skew symmetric matrix.

We need to show that $B^T = -B$, that is, that

$$(A - A^T)^T = -(A - A^T).$$

But $(A - A^T)^T = A^T - (A^T)^T = A^T - A$, and $-(A - A^T) = -A + A^T$, which are equal. Q.E.D.

Problem 7. [18; 3 points each part] True or false. Just write the word "true" or the word "false". If it's not clear to you which it is, explain; otherwise no explanation is necessary.

a. Dot product is commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. True.

b. Matrix multiplication is commutative: $AB = BA$. False.

c. Matrix multiplication distributes over addition: $A(B + C) = AB + AC$. True.

d. Matrix inversion distributes over addition: $(A + B)^{-1} = A^{-1} + B^{-1}$. False. It's false for numbers and it's false for matrices.

e. Matrix inversion distributes over multiplcation: $(AB)^{-1} = A^{-1}B^{-1}$. False, but $(AB)^{-1} = B^{-1}A^{-1}$.

f. Matrix inversion satisfies $(A^{-1})^{-1} = A$. True.