

Math 130 Linear Algebra

Prof. D. Joyce, Clark University

Monday, 21 Sep 2009

Due today. Exercises from section 1.5: 1–2, 5–6, 15–17.

Due Wednesday. Exercises from section 1.6: 1–8, 13–14, 20, 28, T8.

Due Friday. Exercises from section 1.7: 1–4, 9, 11, 12, 17a, 22, T4, T5, ML1–ML4.

Last time. Solutions of linear systems of equations.

Applications. We'll finish 1.7 on inverses of matrices today. That will be the last section of chapter 1 we'll cover. Right after chapter 1, we'll go on to chapter 3 on determinants. We'll look at some of the applications in chapter 2, but not immediately. I'll ask you which section(s) look most interesting. Here's a list of topics from that chapter.

2.1. Error detection/correction codes. These are used in computer science. One that you may have heard of is a parity bit.

2.2. Graphs. The kind of graph under discussion has a set of vertices (also called nodes) and directed edges between them (also called arrows). They're used in mathematics, social sciences, and computer science.

2.3. Computer graphics. Continues the discussion we started in section 1.5 on transformations of the plane.

2.4. Electrical circuits. Kirchhoff's laws for circuits.

2.5. Markov chains. Statistical models where there are several possible states, and at each time

unit, the given state will change according to a probability matrix to one of the other states.

2.6. Linear economic models. Leontief models and others that describe economic transformations in terms of matrices. Analogous to the Markov chains in 2.5, but not probabilistic.

2.7. Wavelets for data compression. The JPEG format used to store images and transmit them across the internet uses this compression scheme.

For next time. Look through chapter 2 to see which topics look interesting. Start reading chapter 3.

Today. Inverse matrices. The concept, the definition, and theorems about inverse matrices. How to compute inverses of matrices. First, 2×2 matrices by a method that uses determinants. We'll talk about determinants in chapter 3 and see how they can be used to find inverses of larger square matrices. Next, we'll find a practical method for finding inverses. Next time we'll see why that method works.

Inverse matrices. As you know, matrix multiplication is unusual. It's not commutative, and cancellation doesn't work. There is no operation of division. But for some square matrices, inversion does work. This is the multiplicative inverse, what we would call reciprocation if we were talking about numbers.

Definition 1. We say that two square $n \times n$ ma-

trices A and B are inverses of each other if

$$AB = BA = I,$$

and in that case we say that B is an inverse of A and that A is an inverse of B . If a matrix has no inverse, it is said to be *singular*, but if it does have an inverse, it is said to be *nonsingular*.

Theorem 2. A matrix can have at most one inverse.

And here's the proof.

Proof. The square matrix A is given. We need to show that there can be at most one matrix B such that $AB = BA = I$. A typical way to prove that there can be at most one of something is to assume that there are two of them, then show they're equal. That's what we'll do here. So, we'll assume that there are two matrices inverse to the matrix A . We need names for them. How about B and C ? That means that we assume that the equations $AB = BA = I$ and $AC = CA = I$ both hold. From them, and from what we already know about matrices, we need to show that $B = C$. If we can, then we conclude that A has at most one inverse.

You can construct a proof by playing with the equations $AB = BA = I$ and $AC = CA = I$. It turns out that addition and subtraction are not helpful operations here, but all you need is multiplication. After you've found a proof, you can probably convert it to an equation, but you might not find an equational proof to begin with. Let's take the first equation $AB = BA = I$ and multiply it by C . We have to be careful with matrix multiplication since it's not commutative. We have to say on which side we're multiplying C . If we multiply on the left, we'll get $CAB = CBA = CI$, but if we multiply on the right we'll get $ABC = BAC = IC$. In this case, either one will do, so let's take the first, $CAB = CBA = CI$. To be very careful about it, we should put in the parentheses, so we should have $C(AB) = C(BA) = CI$. But we know matrix multiplication is associative, and that means the parentheses aren't necessary. Now, what can we do

with the equation $CAB = CBA = CI$? We know I is the multiplicative identity, so we can simplify that to $CAB = CBA = C$. Aha! We've got a C on the right hand side of the equation. Can we get a B on the left hand side? Does CAB simplify to B ? Yes, since $CA = I$, therefore $CAB = IB = B$. Therefore, $B = C$.

Now we've found the proof. The proof needs some cleaning up to be presentable. Here's a summary: $B = IB$ since I is the multiplicative identity; $IB = CAB$ since $I = CA$; $CAB = CI$ since $AB = I$; and $CI = C$ since, again, I is the multiplicative identity. Therefore, $B = C$. Q.E.D.

Finding a proof can take a while. Once you've found it and cleaned it up, it's much shorter.

Now that we know a matrix can have at most one inverse, we're justified in giving a notation for that inverse. We'll use the notation A^{-1} for the inverse of a matrix A , if that inverse exists.

Inverses of inverses. $(A^{-1})^{-1} = A$. The inverse of an inverse is the original matrix. In more detail, if A is invertible, then A^{-1} is also invertible, and the inverse of the matrix A^{-1} is A .

That statement follows directly from the definition. The definition said that B is an inverse of A when $AB = BA = I$. But the equation $AB = BA = I$ is symmetric in A and B , so it also means A is an inverse of B . Thus, B is an inverse of A if and only if A is an inverse of B . Another way of saying that is that the inverse of the inverse of A is A itself. As an equation, that is just $(A^{-1})^{-1} = A$.

Inverses of products. $(AB)^{-1} = B^{-1}A^{-1}$. Note that the order of multiplication is reversed by inversion. In more detail, this theorem says that if both A and B are invertible, then so is AB , and the inverse of AB is the matrix $B^{-1}A^{-1}$. All we have to do is check that $B^{-1}A^{-1}$ actually is the inverse of AB . Are their products in both orders equal to I ? That is, are both $(B^{-1}A^{-1})(AB)$ and $(AB)(B^{-1}A^{-1})$ equal to I ? Yes. First of all, we can

drop the parentheses since multiplication is associative. Now, are both $B^{-1}A^{-1}AB$ and $ABB^{-1}A^{-1}$ equal to I ? Yes, $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$, and $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$. So the inverse of AB is $B^{-1}A^{-1}$.

Inverses of transposes. If a matrix A is invertible, then the inverse of its transpose is the transpose of its inverse, $(A^T)^{-1} = (A^{-1})^T$. To prove this, all you have to do is show that A^T and $(A^{-1})^T$ are inverses, that is, their products in both orders equal I .

Inverses of 2×2 matrices. You can easily find the inverse of a 2×2 matrix. Consider a generic 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

It's inverse is the matrix

$$A^{-1} = \begin{bmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{bmatrix}$$

where Δ is the determinant of A , namely

$$\Delta = ad - bc,$$

provided Δ is not 0. In words, to find the inverse of a 2×2 matrix, (1) exchange the entries on the major diagonal, (2) negate the entries on the minor diagonal, and (3) divide all four entries by the determinant.

You'll verify that this works in a homework exercise.

A method for finding inverse matrices. Next we'll look at a different method to determine if an $n \times n$ square matrix A is invertible, and if it is what it's inverse is. The method is this. First, adjoin the identity matrix to its right to get an $n \times 2n$ matrix $[A|I]$. Next, convert that matrix to reduced echelon form. If the result looks like $[I|B]$, then B is the desired inverse A^{-1} . But if the square matrix in the left half of the reduced echelon form is not the identity, then A has no inverse.

An example. First, let's illustrate the method with a 3×3 example. Let A be the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Form the 3×6 matrix $[A|I]$, and row reduce it. I'll use the symbol \sim when a row-operation is applied. Here are the steps.

$$\begin{aligned} [A|I] &= \left[\begin{array}{ccc|ccc} 3 & -2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & -2 & 4 & 1 & 0 & 0 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 1 & -3 & 0 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -3 & 2 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 3/2 & -1 \end{array} \right] = [I|A^{-1}] \end{aligned}$$

This row-reduction to reduced echelon form succeeded in turning the left half of the matrix into the identity matrix. When that happens, the right half of the matrix will be the inverse matrix A^{-1} . Therefore, the inverse matrix is

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -1/2 & 3/2 & -1 \end{bmatrix}.$$

The question now is: why does this method work? We'll see why next time.

MATLAB can compute inverses or tell you if they're singular.

```
>> A = [1 2; 3 4]
```

```
A =
```

```
1 2
3 4
```

```
>> B = inv(A)
```

```
B =
-2.0000  1.0000
 1.5000 -0.5000
```

```
>> A*B
```

```
ans =
 1.0000  0
 0.0000  1.0000
```

```
>> C = [1 2; 3 6]
```

```
C =
 1 2
 3 6
```

```
>> D = inv(C)
```

Warning: Matrix is singular to working precision.

```
D =
 Inf  Inf
 Inf  Inf
```