Math 130 Linear Algebra

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Due today. Exercises from section 1.7: 1–4, 9, 11, 12, 17a, 22, T4, T5, ML1–ML4.

Due Wednesday. Exercises from section 3.1. Do parts a and b only of all of these exercises: 1-6, 15, 19, and 20.

First test. Select date.

Next time. We'll finish section 3.1 on determinants.

Today. Decide on which sections of chapter 2 to discuss, probably two of them. Begin discussion of determinants in chapter 3.

Introduction to determinants. Every square matrix A has a determinant, denoted either $\det(A)$ or more commonly $|A|$, which is a number that tells a lot about it. We'll see, for instance, that A is an invertible matrix if and only if $|A| \neq 0$. Also, the determinant tells what the transformation described by A does to area. Specifically, the absolute value of the determinant tells you by what factor any region is enlarged.

Determinants of small matrices. Before looking at the general definition for $n \times n$ square matrices, we'll look at the cases when n is small, namely, 2 or 3.

Let A be the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we define the determinant of A to be

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ &= ad - bc. \end{aligned}$$

So, the determinant of a 2×2 matrix is the product of the two elements on the major diagonal minus the product of the two elements on the minor diagonal. We've already seen an application of determinants before in the computation of the inverse of a 2×2 matrix.

Now let A be a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Then we define the determinant of A to be $\det(A) = |A| =$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi.$$

There are six terms in this determinant. Each term is a product of three elements, one element chosen out of each row and column. All six possible ways of choosing one element out of each row and column are included. Three have minus signs and three have plus signs.

Historical note. It's very interesting that determinants *predate* matrices. The study of determinants was going well before anyone had any use for matrices. Indeed, determinants were one of the primary reasons why the theory of matrices was created.

Permutations. It's necessary to know a little about permutations to generalize determinants to $n \times n$ matrices. In fact, permutations are needed just to explain why minus signs are in front of half the terms in the expressions for determinants of 2×2 and 3×3 matrices.

A *permutation* of the set $\{1, 2, 3, \ldots, n\}$ is a list of its n elements where each element appears exactly once in the list. For instance, the six permutations of the set $\{1, 2, 3\}$ are 123, 132, 213, 231, 312, and 321. In general, there are $n!$ permutations of a set of size n. This expression $n!$, read n factorial, is the product of the integers from 1 through n.

An *inversion* occurs in a permutation whenever a larger number appears somewhere before a smaller number. For instance, the permutation 123 has no inversions, but the permutation 312 has two inversions (since 3 precedes 1 and also 3 precedes 2).

The *parity* of a permutation is defined to be either *even* or *odd* if it has an even number or an odd number of inversions, respectively. Of the six permutations of $\{1, 2, 3\}$, exactly three of them are even—123, 231, and 312—and exactly three of them are odd—321, 213, and 132.

Permutation matrices. One way to look at a permutation is to treat it as a matrix itself. First, think of the permutation as an operation rather than a list. For instance, the permutation $\sigma = 21453$ describes an transformation (a function) on the set $\{1, 2, 3, 4, 5\}$ where σ sends 1 to 2, 2 to 1, 3 to 4, 4 to 5, and 5 to 3. Associate this to the 5×5 matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

This matrix then operates on a column vector as a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \\ a_4 \\ a_5 \\ a_3 \end{bmatrix}$$

Thus, the permutation matrix permutes the rows on the right. The row 1 is replaced by row 2, row 2 by row 1, row 3 by row 4, row 4 by row 5, and row 5 by row 3.

Permutations and matrices. When we define the determinant of a square $n \times n$ matrix, which we'll do in a moment, it will be defined as a sum/difference of $n!$ terms, each term being a product of n elements, one element chosen out of each row and column. Our first question is: whey are there $n!$ ways to choose one element out of each row and column? Each choice is determined by which column to choose for each row. So, if the element a_{1j_1} is chosen for the first row, a_{2j_2} is chosen for the second row, …, and a_{nj_n} is chosen for the nth row, that choice is determined by the numbers $j_1 j_2 \ldots j_n$. But $j_1 j_2 \ldots j_n$ is just a permutation of the set $\{1, 2, \ldots, n\}$. So, there are $n!$ choices, and each corresponds to a permutation.

Definition of determinant. We can now define the determinant of an $n \times n$ matrix. Form all $n!$ products of n elements, one element chosen out of each row and column. Negate all those that correspond to odd permutations, but don't negate those that correspond to even permutations. Add all these $n!$ terms together. That's the determinant.

This definition usually is used to compute determinants when n is small, 2 or 3, and it agrees with what we did above. But when n is 4 or greater, there are so many terms that it isn't practical to use the definition to compute the value of a determinant. We'll see faster ways of computing determinants soon.

The determinant of a 4×4 matrix. Let's take a generic matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Look at all $4! = 24$ permutations of the set $\{1, 2, 3, 4\}$ and their parities. Even parities are indicated with $+$, odd with $-$.

1234	+	2134	−	3124	+	4123	−
1243	−	2143	+	3142	−	4132	+
1324	−	2314	+	3214	−	4213	+
1342	+	2341	−	3241	+	4231	−
1423	+	2413	−	3412	+	4312	−
1432	−	2431	+	3421	−	4321	+

Each entry in the above table gives one term in the determinant of A. Thus, reading down the first column, we see that the determinant starts out with the following six terms:

$$+a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44}$$
$$+a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42}$$

But besides these, there are 18 more terms.

MATLAB can find determinants. Let's see what it gets for one using the `det` function.

```
>> A=[1,3,5,7;2,4,8,6;0,1,5,3;1,1,0,0]

A =
     1     3     5     7
     2     4     8     6
     0     1     5     3
     1     1     0     0

>> det(A)

ans =
    -2
```

An interesting matrix is the Hilbert matrix generated in MATLAB using the `hilb` function. You can change the way numbers are displayed as described in section 12.1.

```
>> H = hilb(5)

H =
 1.0000 0.5000 0.3333 0.2500 0.2000
 0.5000 0.3333 0.2500 0.2000 0.1667
 0.3333 0.2500 0.2000 0.1667 0.1429
 0.2500 0.2000 0.1667 0.1429 0.1250
 0.2000 0.1667 0.1429 0.1250 0.1111

>> format rat
>> H

H =
 1     1/2   1/3   1/4   1/5
 1/2   1/3   1/4   1/5   1/6
 1/3   1/4   1/5   1/6   1/7
 1/4   1/5   1/6   1/7   1/8
 1/5   1/6   1/7   1/8   1/9

>> det(H)

ans =
       1/266716800000
```

That's one small determinant!