Math 130 Linear Algebra

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Due today. Exercises from section 4.3: 25, 26, 27, 29, T9, T10, T11.

Due Friday. Exercises from section section 5.1: 1ab, 2ab, 9-12, T2, T3, T4, T5, (T7), ML1–2, ML5–6.

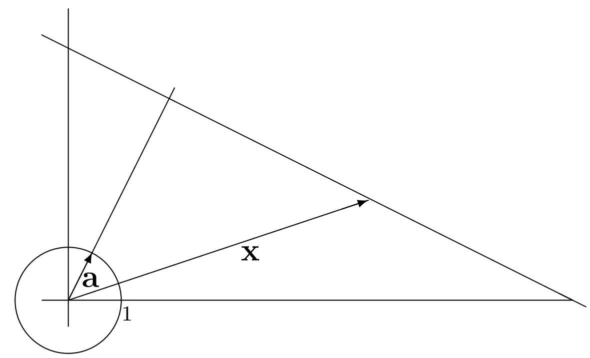
Due Monday. Exercises from section 5.2: 5ab, 6ab, 10ab, 13, T3.

Last time. The cross product and triangles, parallelograms, and parallelepipeds.

For next time. Read section 6.1 on abstract vector spaces.

Today. We'll discuss section 5.2 on lines and planes.

A vector equation for lines in the plane, and planes in space. Every linear equation in x and y describes a line in the plane $\mathbf{R}^2$, and conversely, every line can be described by such a line. A typical line has the equation $ax+by=c$, where a, b, and c are constants. We recognize the left hand side of the equation as the dot product of the two vectors $\mathbf{a} = (a,b)$ and $\mathbf{x}=(x,y)$. So this equation says $\mathbf{a}\cdot\mathbf{x}=c$. We can normalize that equation by dividing the constants a, b, and c by $\|\mathbf{a}\|$ so that we may assume that $\mathbf{a}$ is a unit vector.



In general, $\mathbf{a}\cdot\mathbf{x} = \|\mathbf{a}\|\,\|\mathbf{x}\|\cos\theta$, where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{x}$, but when $\mathbf{a}$ is a unit vector, that equation simplifies to $\mathbf{a}\cdot\mathbf{x} = \|\mathbf{x}\|\cos\theta$. We can interpret this quantity as a distance as follows. Project the vector $\mathbf{x}$ on to the unit vector $\mathbf{a}$. We get a right triangle with one side being that projection, and the hypotenuse $\mathbf{x}$. The projection has length $\|\mathbf{x}\|\cos\theta$. Thus , $\mathbf{a}\cdot\mathbf{x}$ is the length of the projection of $\mathbf{x}$ on to $\mathbf{a}$.

Now we can interpret the vector equation

$$\mathbf{a}\cdot\mathbf{x} = c$$

as saying the point $\mathbf{x}$ lies on the line orthogonal to the vector $\mathbf{a}$ at a distance c from the origin, at least in the case when $\mathbf{a}$ is a unit vector. A vector $\mathbf{a}$ orthogonal to the plane is also said to be *normal* to the plane. (It is unclear why we need so many words—perpendicular, orthogonal, and normal—that all mean the same thing.)

When we move up a dimension, the equation $\mathbf{a}\cdot\mathbf{x}=c$ describes the plane in $\mathbf{R}^3$ orthogonal to the vector $\mathbf{a}$, and when $\mathbf{a}$ is a unit vector, then the plane is at a distance c from the origin.

A determinant for describing the line in $\mathbf{R}^2$ determined by two points. Suppose $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are two points on a line. What is an equation for that line? Let $\mathbf{x} = (x_1, x_2)$ be any point on that line. We know that in general, the absolute value of the determinant

$$\begin{vmatrix} x_1 & x_2 & 1 \\ a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \end{vmatrix}$$

describes the area of a parallelogram, three of whose vertices are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{x}$. The area will be 0 if and only if those three vectors line a a line. Thus, the equation

$$\begin{vmatrix} x_1 & x_2 & 1 \\ a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \end{vmatrix} = 0$$

describes the line passing through the two points $\mathbf{a}$ and $\mathbf{b}$. Note that the equation has just the two real variables x_1 and x_2.

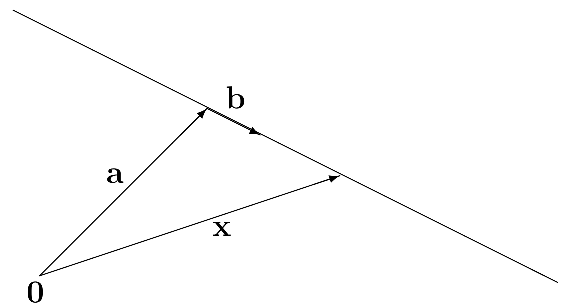
A determinant for describing the plane in $\mathbf{R}^3$ determined by three points. Of course, any three noncollinear points $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_3)$ in $\mathbf{R}^3$ determine a plane in $\mathbf{R}^3$. A point $\mathbf{x} = (x_1, x_2, x_3)$ lies on that plane if and only if

$$\begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{vmatrix} = 0$$

Parametric descriptions for lines in the plane and in space. An alternate way to describe a line, either in the plane or in space, is to name one point $\mathbf{a}$ on the line and say that other points on the line can be found by going some distance in a direction $\mathbf{b}$ from that point. Then a typical point on the line is of the form

$$\mathbf{x} = \mathbf{a} + t\mathbf{b}$$

where t is any real number. Here, we say that t is a parameter for describing points on the line.



This parametric equation actually describes a function $\mathbf{R} \to \mathbf{R}^n$ from the line $\mathbf{R}$ to n-space.