Math 130 Linear Algebra Quiz, Oct 2009

Scale. 9–10 A, 7–8 B, 5–6 C. Median 8.

Problem 1. [6] Computations.

a. Find a unit vector in the direction of $(4, 1, 2, 2)$.

Divide the vector by its length. Its length is

$$\|(4, 1, 2, 2)\| = \sqrt{4^2 + 1^2 + 2^2 + 2^2} = \sqrt{25} = 5$$

so the unit vector is

$$\frac{(4, 1, 2, 2)}{5} = (\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}).$$

b. Name a nonzero vector orthogonal to the vector $(2, 3, 1)$.

We need to find a vector (x, y, z) whose dot product with $(2, 3, 1)$ is 0, that is, we need to solve the equation

$$2x + 3y + z = 0.$$

There are many solutions, for instance, $(1, 0, -2)$.

c. Determine the area of the triangle with vertices $(1, 2)$, $(3, 4)$, and $(5, -6)$. Note that the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 5 & -6 & 1 \end{bmatrix}$$

is -24.

The area is 1/2 of the absolute value of that determinant, 12.

Problem 2. [4] Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in the plane. The law of cosines implies that

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\,\cos\theta$$

where θ is the angle beween the two vectors. Use that equation to prove that

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\,\cos\theta.$$

You may use the definitions and/or properties of length and dot product in your proof.

There are at least two different approaches to a proof. One uses coordinates for the vectors while the other doesn't. The one without coordinates is easier, so here's one without coordinates. It hinges on the property that the square of the length of a vector is the dot product of that vector with itself.

Proof. Since it's given that

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\,\cos\theta$$

it will follow that

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\,\cos\theta$$

if we can show that

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\mathbf{v} \cdot \mathbf{w}.$$

But the left side of that equation is

$$\begin{aligned} \|\mathbf{v} - \mathbf{w}\|^2 &= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \end{aligned}$$

while the right side of that equation is

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - 2\,\mathbf{v} \cdot \mathbf{w}$$

which are equal since dot products are commutative. Q.E.D.